Numerical Study on Plasma Flowfield and Performance of Magnetoplasmadynamic Thrusters

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January, 2009
Acknowledgements

I would like to express my gratitude to Prof. Yoshihiro Okuno of Tokyo Institute of Technology for providing me with the opportunity to research at his laboratory. His considerable insights on plasma science have been supported my studies of master and doctoral courses.

I am also deeply grateful to Associate Prof. Ikkoh Funaki of ISAS/JAXA for accepting me as a research fellow at ISAS. He taught me the electric propulsion technology and the computational techniques from the beginning, and generously gave me a lot of invaluable advice on my research. I have learned attitude toward research activities from him.

I would like to thank Prof. Hiroyuki Yamasaki, Prof. Tetsuji Okamura, and Assistant Prof. Tomoyuki Murakami for their fruitful advice for my research in MHD group meeting. I also want to thank Dr. Kumiko Ogaki and Engineering Official Hiroshi Takahashi for their helpful activities in our group.

I greatly appreciate the valuable instructions by Dr. Daisuke Nakata and Dr. Hiroshi Katsurayama of ISAS/JAXA.

I am grateful to the members in MHD group of Tokyo Institute of Technology and Abe/Funaki laboratory of ISAS/JAXA, especially to my precious colleagues Mr. Masaharu Matsumoto and Mr. Keisuke Udagawa for the energetic discussion on researches. Also, I want to thank Ms. Yumi Usui for helpful works on my research.

Finally, I would like to express deepest appreciation to my family for considerate support.
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Nomenclature

\( A, A = \) vector potential, \( T \cdot m^{-1} \)
\( B, B = \) magnetic flux density, \( T \)
\( C = \) thermal velocity, \( m \cdot s^{-1} \)
\( e = \) elementary charge, \( C \)
\( E = \) electric field, \( V \cdot m^{-3} \)
\( \delta E = \) energy exchange via collision, \( W \cdot m^{-3} \)
\( \Delta E = \) input energy, \( J \)
\( F = \) thrust, \( N \)
\( g = \) acceleration of gravity, \( m \cdot s^{-2} \)
\( h = \) Planck constant, \( J \cdot s \)
\( I_{sp} = \) specific impulse, \( s \)
\( j, j = \) current density, \( A \cdot m^{-2} \)
\( J = \) total discharge current, \( A / Jacobian \)
\( k = \) Boltzmann constant, \( J \cdot K^{-1} \)
\( l = \) mean free path, \( m \)
\( M = \) Mach number
\( m_s = \) mass of a particle \( s \), \( kg \) (\( m_i = m_n = m_i \) is assumed.)
\( M' = \) atomic weight, \( amu \)
\( \dot{m} = \) mass flow rate, \( kg \cdot s^{-1} \)
\( \Delta m = \) mass shot, \( kg \)
\( n = \) number density, \( kg \cdot m^{-3} \)
\( p = \) pressure, \( Pa \)
\( P = \) input power, \( W \)
\( Pr = \) Prandtl number
\( Q_{ij} = \) collision cross section between particles \( i \) and \( j \), \( m^2 \)
\( r = \) radial coordinate, \( m \)
\( R_m = \) Magnetic Reynolds number
\( T = \) temperature, \( K \)
\( T_{on} = \) discharge duration time, \( s \)
\( t = \) time, \( s \)
\( u, u = \) velocity, \( m \cdot s^{-1} \)
\( U = \) internal energy per unit volume, \( J \cdot m^{-3} \)
\( U_i = \) ionization energy per unit volume, \( J \cdot m^{-3} \)
\( V_i = \) ionization energy from \( i-1 \) to \( i \)-fold ionized species, \( eV \)
\( V = \) voltage, \( V \)
\( W = \) thruster width of 2-dimensional MPD thruster, \( m \)
\( z = \) axial coordinate, \( m \)
Greek Symbols

$\alpha$ = ionization fraction
$\beta$ = Hall parameter / plasma beta
$\gamma$ = specific ratio (=5/3)
$\varepsilon_0$ = permittivity in vacuum, F·m$^{-1}$
$\eta$ = efficiency / resistivity, $\Omega$·m / generalized coordinate
$\theta$ = azimuthal coordinate
$\lambda$ = thermal conductivity, W·m$^{-1}$·K$^{-1}$
$\tilde{\lambda}_D$ = Debye length, m
$\mu$ = viscosity coefficient, N·s·m$^{-2}$
$\mu_0$ = permeability in vacuum, H·m$^{-1}$
$v_{ij}$ = collision frequency between particles i and j, s$^{-1}$
$\xi$ = generalized coordinate
$\rho$ = mass density, kg·m$^{-3}$
$\sigma$ = electrical conductivity, $\Omega^{-1}$·m$^{-1}$
$\tau$ = viscous tensor, N·m$^{-2}$
$\tau$ = discharge time constant, s
$\Phi$ = dissipation function, W·m$^{-3}$
$\psi$ = stream function for current, T·m
$\omega$ = frequency, s$^{-1}$

Subscripts/Superscripts

$\text{a}$ : anode
$\text{ac}$ : acoustic
$\text{an}$ : anomalous
$\text{ap}$ : applied
$\text{b}$ : backward / boundary
$\text{c}$ : critical / cathode / cyclotron / characteristic
$\text{coil}$ : coil
$\text{e}$ : electron
$\text{ex}$ : exhaust / excitation
$\text{f}$ : forward
$\text{h}$ : heavy particle (neutral and ion)
$\text{i}$ : ion / ionization
$\text{ind}$ : induced
$\text{L}$ : Larmor
$\text{max}$ : maximum
$\text{n}$ : neutral / normal
$\text{rad}$ : radiation
$\text{s}$ : species / shock
$\text{sh}$ : sheath
$\text{th}$ : thermal
Remarks on Units

The MKSA system is basically used for variables in equations. When CGS unit is exceptionally used, a note of caution is added. In the text, for simple notation, electron-volt (1 eV = 11,600 K) is often used for temperature.
Chapter 1.

Introduction

In recent years, advances of space technologies in 20th century have been spreading areas of space missions from the near-Earth sphere to further astronomical objects. In addition to space explorations, the construction of the international space station will finish within some years, which impresses dawn of a new space age on us. Magnetoplasmodynamic thrusters discussed in this thesis are one of electric propulsion devices which will become the key technology for a mission requiring a high velocity increment in the next generation. In this chapter, electric propulsion technologies are outlined, and subsequently, general properties and preceding studies of magnetoplasmodynamic thrusters are shown. At the end, motivations and objectives of this study are described.

1.1 Electric Propulsion Technologies

1.1.1 Rocket Equation

A space mission to a distant astronomical object and orbital transfer of a large structure entails a high velocity increment $\Delta V$. To obtain large $\Delta V$, a high exhaust velocity $u_{ex}$ of propellant is desired for spacecraft as shown in the rocket equation derived by Tsiolkovsky [1]:

$$\Delta V = u_{ex} \ln \frac{M_i}{M_f} = u_{ex} \ln \frac{M_i}{M_i - M_{pp}}$$  \hfill (1.1)
Chapter 1 Introduction

with the definitions of

\[ M_i = M_{pl} + M_{pd} + M_{pwr} + M_{pp}, \quad M_f = M_{pl} + M_{pd} + M_{pwr}. \]

Here, \( M_i \), \( M_f \), and \( M_{pp} \) denote an initial and final mass of a spacecraft, and a propellant mass respectively. The initial mass \( (M_i) \) includes the masses of payload \( (M_{pl}) \), propulsion device \( (M_{pd}) \), power source \( (M_{pwr}, \ M_{pwr} = 0 \) for chemical propulsion), and propellant \( (M_{pp}) \). There are two categories for spacecraft propulsion; chemical propulsion, and electric propulsion (Fig. 1.1). A chemical propulsion device generally obtains a thrust via energy conversion from chemical energy released by combustion to thrust power. Since \( u_{ex} \) for a chemical propulsion device is restricted by chemical energy stored in their propellant, a large ratio of \( M_i/M_f \) is needed to attain \( \Delta V \), which corresponds to a large amount of propellant mass \( (M_{pp}) \). On the other hand, a thrust power for an electric propulsion device originates from an electric power input generated from the solar panels. An electric propulsion device accelerates a propellant by some electric effects, thus its \( u_{ex} \) can be widely varied by changing applied electric power and/or acceleration mechanism. Consequently, an electric propulsion system can obtain higher \( u_{ex} \) than that of chemical propulsion by optimizing operation conditions, resulting in small amount of propellant mass \( (M_{pp}) \).

Specific impulse \( (\equiv u_{ex}/g, \ g: \) acceleration of gravity), which will be defined in the next subsection, is limited to about 450 s for a chemical propulsion device, but can be increased up to 1,000-10,000 s by using an electric propulsion device.

Fig. 1.1 Concepts of chemical and electric propulsion.
1.1.2 Parameters for Performance Evaluation

Typical parameters to assess the performance of a thruster are thrust $F$, specific impulse $I_{sp}$, and thrust efficiency $\eta$ defined as follows:

$$ F = \dot{m}u_{ex}, $$

$$ I_{sp} = \frac{F}{mg} = \frac{u_{ex}}{g}, $$

$$ \eta = \frac{\dot{m}u_{ex}^2/2}{P} = \frac{F^2}{2mP} = \frac{g}{2}I_{sp}F/P. $$

The ratio $F/P$ is called as thrust-power ratio. From Eq. (1.3), high $u_{ex}$ is equivalent to high $I_{sp}$, which corresponds to a fuel-efficient thruster. Under constant $F$ and $\eta$, however, Eq. (1.4) indicates that too high $I_{sp}$ leads to a detrimental increase in input power, thus it is necessary to find the optimal $I_{sp}$ for a mission. For a given $\Delta V$, the optimal $I_{sp}$ for the maximum payload ratio can be approximated by the following equation [2]:

$$ I_{sp,opt} \equiv \sqrt{\frac{\beta t}{g}} \equiv \frac{u_{ch}}{g}, $$

where

$$ \beta = \frac{2\eta}{\alpha_{pd} + \alpha_{pwr}}, \quad \alpha_{pd} = \frac{M_{pd}}{P}, \quad \alpha_{pwr} = \frac{M_{pwr}}{P}. $$

Here, $t$ represents mission time. On the other hand, it is known that, for a given payload ratio, the optimal $I_{sp}$ for the maximum velocity increment $\Delta V$ can be approximated by the same equation as Eq. (1.5), thus the maximum payload ratio and the maximum $\Delta V$ are reached at the same time [3]. Assuming that the input energy during a mission time $t$ is completely converted into the kinetic energy of a propulsion device and a power source:

$$ \frac{1}{2}(M_{pd} + M_{pwr})u_{ch}^2 = \eta Pt. $$

From Eq. (1.6), it can be said that $u_{ch}$ corresponds to the maximum attainable velocity of a spacecraft without payload. We have to pay attention to the fact that an important indicator of overall ability of a propulsion system is not thrust efficiency $\eta$ but $u_{ch}$ [4]. It is noted that Eq. (1.5) insists that a reduction of a specific mass of a propulsion device $\alpha_{pd}$ and/or a power source $\alpha_{pwr}$ is equivalent to an increase in thrust efficiency $\eta$. 
1.1.3 Thruster Types

Acceleration mechanisms of electric propulsion devices are classified into three categories of \textit{electrothermal}, \textit{electrostatic}, and \textit{electromagnetic} modes. Electrothermal types aerodynamically accelerate a propellant heated by an electric heater (Resistojet) or by Joule heating with arc discharge (DC-Arcjet) through a solid nozzle. Since heating rate is restricted by the melting temperature of walls, an attainable specific impulse is limited by about 1,000 s. In the case of electrostatic types (Ion thruster, Field emission electric propulsion, Hall thruster), a propellant ionized by discharge is accelerated by an electrostatic potential drop toward a downstream region. Normally these thrusters’ high thrust efficiency is attributed to their low thermal and ionization losses. Regarding the electromagnetic types, discharge plasma is accelerated mainly by Lorentz force as well as by an aerodynamic expansion. The production mechanism of the Lorentz force classifies the electromagnetic thrusters into several types such as magnetoplasmadynamic thruster (MPDT), pulsed plasma thruster (PPT), and Hall thruster which is a hybrid type combining electrostatic and electromagnetic acceleration mechanisms. Typical working conditions and performances of the above thrusters except for MPDTs are tabulated in Table 1.1 [5]. Most of the thrusters shown here have been used for some practical purposes, but operations of MPDTs are limited to experimental flights due to its low thrust efficiency at the present stage.

The feature of high \(I_{sp}\) have been encouraged adoption of electric propulsion for a variety of missions. Main applications have been attitude control, geosynchronous stationkeeping, and orbit adjustments until 1999, and recently they came to a main propulsion system of a spacecraft (Table 1.2). Furthermore, desirable power range is now getting larger in accordance with expanding scales of space missions from several kW to several tens of kW. Larger thrust requirements for many missions requisitely led to the concepts of clustering Hall thruster [10] and 50 kW level Hall thruster [11]. In addition to Hall thrusters, MPDTs are candidate for electric propulsion devices capable of high power operation from 10 kW up to 1 MW.
### 1.2 Magnetoplasmadynamic Thrusters

#### 1.2.1 General Characteristics of MPD Thrusters

An MPDT is one of high-power electric propulsion devices for an interplanetary spacecraft, and is classified into two categories; self-field MPDT (SFMPDT), and applied-field MPDT (AFMPDT). In terms of operation modes, there will be steady or pulsed operations for SFMPDT and AFMPDT respectively. The principles of SFMPDT and AFMPDT are illustrated in the following Subsec. 1.2.2 and 1.2.5 respectively. Before the explanations on the detailed principles of MPDTs, introduction of their general characteristics will be helpful to specify the status of this study. Basically, MPDTs have following advantages compared to the other thrusters.

1. high thrust density (~\(10^4\) N/m\(^2\))
2. capability with high power
   - (SFMPDT: 100kW ~ MW, AFMPDT: 10kW ~ 100kW)
3. simple structure
4. diversity of available propellants (Ar, He, H\(_2\), NH\(_3\), N\(_2\)H\(_4\) etc.)

The first and the third features contribute to the reduction of the mass of a thruster-head. In contrast to Ion thrusters in which thrust density is restricted by space-charge limited...
current, the thrust density of MPDTs can be widely adjusted by varying input power. Generally, the feature of light thruster-head reduces the specific mass of a propulsion device $\alpha_{pd}$, thus the required thrust efficiency to obtain comparable $u_{ch}$ (Eq. (1.6)) to the Ion or Hall thrusters is estimated to be 40 - 45% [4]. According to experimental results, this can be achieved by using H$_2$ or Li propellant, although actual application of these propellants encounters difficulties with regard to a storage method and/or contamination of a spacecraft. With other propellants such as Ar, He etc., attainable thrust efficiency is lower than the target at this present stage. The improvement of thrust efficiency is one of the most important issues for MPDTs.

Equation (1.4) can be rewritten as

$$P = \frac{g I_p F}{2 \eta},$$

(1.7)

which indicates that a large input power $P$ is necessary to maintain both a large thrust and a high specific impulse under the condition of constant thrust efficiency. Since suitable power of MPDTs ranges from several tens of kW to several MW, MPDTs will be suitable for a deep space exploration in the next generation if thrust efficiency is improved. In particular, as shown in Fig. 1.2, MPDTs is indispensable for a mission operated with electric power above 100 kW for which a nuclear power source will be suitable.

![Fig. 1.2 Performances of electric thrusters [12].](image-url)
1.2.2 Self-Field MPD Thrusters

The principle of a SFMPDT is shown in Fig. 1.3. Generally, a SFMPDT has a coaxial anode surrounding a central cathode. In a thruster, a magnetic field induced by a discharge current is utilized to produce Lorentz force to accelerate an ionized propellant. The axial component of the Lorentz force is called as **blowing force**, which directly accelerates the ionized propellant. On the other hand, the radial Lorentz force contributes to thrust by increasing the pressure around a cathode tip. Surface force acting on a cathode tip is denoted as **pumping force**. Electromagnetic force $F_{em}$ including both blowing and pumping forces can be approximated by the Maecker formula [1]

$$F_{em} = \mu_0 J^2 \left( \ln \frac{r_a}{r_c} + d \right) \equiv b J^2 \quad (0 \leq d \leq 3/4), \quad (1.8)$$

where $r_a$ and $r_c$ are radii of an anode and a cathode respectively, and the parameter $d$ is selected based on current distribution on the cathode surface; the $d$ parameter approaches 3/4 as discharge current concentrates at the cathode tip. This formula indicates that electromagnetic thrust caused by the self-field is proportional to $J^2$, and the proportionality factor $b$ basically depends only on thruster configuration. To obtain adequate strength of the induced magnetic field to accelerate a plasma, required discharge current amounts to around 10 kA, and then discharge voltage of a typical

---

* There is a thrust formula including the effect of aerodynamic acceleration derived by Choueiri, whereas the formula is specialized for their thruster configuration [13].
MPDT \((r_o \approx 5 \text{ cm})\) is 50 – 200 V, hence total electric power ranging from several hundreds of kW to several MW is required.

As illustrated in Fig. 1.3, an MPDT normally has coaxial electrodes. Despite many diagnostic efforts on coaxial MPDTs, severe discharge and perturbations caused by inserted probes had impeded satisfactory measurement of plasma flowfields in a thruster. In the late 1980s, in order to obtain reliable aspects of flowfields, a two-dimensional MPDT (2D-MPDT) in Fig. 1.4 was developed at ISAS (Institute of Space and Astronautical Science, Japan) [14]. Although it is difficult for 2D-MPDT to precisely reproduce features of coaxial MPDTs, two-dimensional plasma data of 2D-MPDT are very fruitful to understand physical processes inside a thruster. Regarding the formula of electromagnetic thrust of a 2D-MPDT, the following expression was derived [15]:

\[
F_{em} = \frac{\mu_0 d_a J^2}{8W}.
\]  

\text{(1.9)}

Here, \(d_a\) and \(W\) represent interval between upper and lower anodes, and width of the thruster respectively.

![Fig. 1.4 Schematic view of a 2D-MPD thruster.](image)

1.2.3 Preceding Studies of Self-Field MPD Thrusters

A variety of experimental and theoretical efforts for performance improvement of SFMPDTs have been reported over the past four decades [16,17]. Experiments of high power SFMPDTs have been conducted mainly in the United States, Europe, and Japan with quasi-steady mode, in which MW power level operation continues for a few
milliseconds. Some of the highest measured efficiencies for coaxial SFMPDTs with various propellants are summarized in Table 1.3 [17]. As for the research of steady-state operation, experiments in the range of 100 – 300 kW have been continued in Germany [18]. For further high power range operation, NASA Glenn Research Center is developing and testing quasi-steady MW-class thrusters as a preliminary step to steady-state high power thruster [19].

In addition to the above mentioned experimental efforts, improved availability of sufficient computational power enables us to investigate detailed plasma flowfields in SFMPDTs. Princeton University performed a full-scale analysis of a benchmark thruster [20], and Stuttgart University simulated DT2 MPDTs comprising a converging-diverging insulator nozzle and a flared anode at the end of the thruster [21]. While the above research projects adopted argon as a propellant, Mikellides focused on MW-class thrusters with hydrogen propellant, and computed a flowfield by using the MACH2 code [22].

<table>
<thead>
<tr>
<th>Propellants</th>
<th>H₂</th>
<th>N₂</th>
<th>NH₃</th>
<th>CH₄</th>
<th>Ar</th>
<th>Ne</th>
<th>He</th>
<th>O₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>η (%)</td>
<td>55</td>
<td>38</td>
<td>35</td>
<td>33</td>
<td>30</td>
<td>20</td>
<td>20</td>
<td>18</td>
</tr>
<tr>
<td>Iₚₑ (s)</td>
<td>10000</td>
<td>4000</td>
<td>5700</td>
<td>6000</td>
<td>2300</td>
<td>4000</td>
<td>2000</td>
<td>2500</td>
</tr>
</tbody>
</table>

1.2.4 Critical Current

According to Eq. (1.8), increasing discharge current under a fixed mass flow rate can enhance thrust of SFMPDTs. This procedure, however, confronts difficulties above a certain level of discharge current, so-called critical current $J_c$, where a discharge mode transition occurs from a diffused mode to a spot mode, and voltage fluctuate as shown in Fig. 1.5 [23,24]. In addition, it is commonly known that an acceleration mode turns into an electromagnetic mode from an electrothermal mode around the critical current. Well above the critical current, operation of MPDTs becomes unstable, accompanied by voltage fluctuations at random frequencies in the order of 1 MHz. This harmful

* A lifetime test of a cathode had been performed for 1 hour with 200 kW.
phenomenon, so-called *onset*, restricts performance of SFMPDTs.* Although detailed mechanism of onset has not been specified despite intensive researches, it is commonly believed that the critical current is approximately given by a current which provides electromagnetic thrust corresponding to a thrust with exhaust velocity $u_c$ [25]

$$\dot{m}u_c = bJ_c^2 \Rightarrow \frac{J_c^2}{\dot{m}} = \frac{u_c}{b} = \frac{1}{b} \sqrt{\frac{2eV_i}{m_i}},$$

(1.10)

where $u_c$ is called as *Alfvén’s critical velocity* [26]. The $b$ value is identical with that in Eq. (1.8). This relation can be derived from the principle of least work. When $P_{flow}$ represents the power deposited in a fully ionized flow,

$$P_{flow} = \frac{1}{2} \dot{m}u_{ex}^2 + eV_i \frac{\dot{m}}{m_i} = \frac{F^2}{2\dot{m}} + eV_i \frac{\dot{m}}{m_i},$$

(1.11)

where specific enthalpy is ignored. Under the condition that an electromagnetic force is dominant, i.e. $F = bJ^2$, when $P_{flow}$ is minimized (minimum discharge voltage) with regard to $\dot{m}$, the following relationship equivalent to Eq. (1.10) can be derived [25]:

$$\frac{\partial P_{flow}}{\partial \dot{m}} = 0 \Rightarrow \frac{F^2}{2\dot{m}} = eV_i \frac{\dot{m}}{m_i}$$

$$\Rightarrow \frac{J_c^2}{\dot{m}} = \frac{1}{b} \sqrt{\frac{2eV_i}{m_i}}.$$  

(1.12)

The second expression indicates that, when the critical current flows, the kinetic energy equals to the ionization energy required for fully ionization. In spite of this simple consideration, the critical current defined by Eq. (1.10) can be a good indicator of performance limitation. To avoid unstable discharge, SFMPDTs should be operated around the critical current. However, stable operation far above the critical current is preferred for the purpose of enhancing thrust performance.

There are several propositions concerning the mechanism of onset. The *back-EMF theory* developed by Lawless and Subramaniam states that choking condition is related with an onset condition, although this theory is quasi-one dimensional and ignores the Hall effect [27-29]. Also, there are suggestions that some *microinstabilities* may induce

---

* A current value resulting in onset (intensive voltage fluctuation) is not necessarily equal to a critical current, and is denoted as ‘limiting current’ in Ref. [23]. Depending on propellant species, a limiting current can be more or less than a critical current estimated by Eq. (1.10) [23].
the onset [30-33]. In addition, several theories focus on plasma conditions near an anode surface, where a radial pinch force can lead to shortage of current-carrier in the context of depletion of plasma density near the anode [18,34]. This phenomenon is called as anode starvation (Fig. 1.6). In recent numerical studies [20,21], depletion of plasma density near the anode is certainly appeared at $J > J_c$, but it is not specified in what condition the anode starvation, that is current-carrier shortage, occurs. A numerical study on current-carrier shortage is limited to the Hügel’s work in which some effects of viscosity and thermal conduction were ignored [34].

In spite of many experimental and theoretical efforts, plasma behavior around the critical current is still under discussion. To overcome the restriction on maximum allowable discharge current, understandings of plasma flowfields at high $J^2/m$ is necessary.

Fig. 1.5 Typical waveforms of discharge voltage, Ar, 2.42 g/s, (a) 8 kA, (b) 12.5 kA, (c) 16.5 kA, (d) 17.2 kA, (e) voltage waveform obtained using a low-pass filter of 20 kHz [23].

Fig. 1.6 Mechanism of anode starvation.
1.2.5 Applied-Field MPD Thrusters

Within an AFMPDT, a magnetic field (~ 0.1 T) is axially applied by an external coil or a permanent magnet surrounding an anode (Fig. 1.7). The applied magnetic field leads to some characteristic phenomena in comparison with flowfields in SFMPDTs. As discharge current crosses the applied magnetic field, azimuthal Lorentz force \( j_r B_z \) drives an azimuthal flow (swirl flow) which is to be converted into an axial flow through a solid/magnetic diverging nozzle. This acceleration mechanism is denoted as *swirl acceleration*. In addition, application of the magnetic field yields an azimuthal induced current \( j_\theta \) mainly by the Hall effect and diamagnetic effect. The azimuthal induced current and the radial component of the applied magnetic field produce axial Lorenz force \(-j_\theta B_r\), which directly increases thrust in axial direction. Also, the azimuthal current and the axial magnetic field cause radial Lorentz force \( j_\theta B_z \), which increases the pumping force to confine plasma. Acceleration with regard to the azimuthal current is called as *Hall acceleration*. Here, it has to be noted that contributions of swirl and Hall acceleration to total thrust cannot be specifically isolated, because momentum conversion process of a swirl flow into an axial flow via a magnetic nozzle entails the Lorentz force with regard to the azimuthal current [35]. Detailed discussion on thrust components is conducted in Subsec. 5.2.5.

Basically, these additional acceleration mechanisms provide larger total thrust than
that of a SFMPDT, thus the discharge current for AFMPDT to achieve a certain thrust level can be reduced by optimizing working conditions. Typical discharge current of an AFMPDT is less than 1 kA, and generally input power can be alleviated down to several tens of kW.

Performances obtained from a variety of experiments on AFMPDTs are well summarized in Ref. [36]. According to the experimental evidences, it has been observed that thrust is proportional to the product of a discharge current and an applied magnetic field, which is generally believed to be due to swirl acceleration [35]. On the other hand, there is still much debate concerning both theoretical and experimental evidence for Hall acceleration. It is commonly supposed that Hall acceleration can be dominant under relatively strong applied magnetic field and a low mass flow rate [37]. Basically increasing external magnetic field enhances thrust efficiency at a constant specific impulse [38], but it was also reported that there is an optimum magnetic field intensity to enhance thrust efficiency [39].

In almost all studies, alkali vapor, especially lithium, or gases (hydrogen, Argon, etc.) have been employed as a propellant. As far as the data obtained with ambient pressure less than 1 mTorr are concerned, the best thrust efficiency is 69% with lithium propellant at 21 kW [40]. In the case of gas-fed operation, the best thrust efficiency with hydrogen is 47% at 3.6 MW [38]. As for the result with argon propellant, 25% at 59 kW is the best thrust efficiency [41]. There is also an experimental result with argon propellant in the lower power range about 20 kW [37].

There are not so many numerical studies on AFMPDTs. Tanaka et al. simulated plasmas in an AFMPDT with two-dimensional and quasi-one dimensional formulations for magnetic field and fluid respectively [42]. Mikellides simulated flowfields in the NASA Lewis Research Center 100-kW-class applied-field MPD thruster by using the two-dimensional simulation code (MACH2) [43]. Recently, development of a simulation code (SAMSA) for 10-20 kW class AFMPDT was reported by Haag [44]. Although basic properties of flowfields have been shown, thrust production mechanisms (thrust component) and energy conversion processes within AFMPDTs have not been revealed yet.
1.2.6 Power Source in Space

Development of 10 kW – MW class electric power source will be indispensable to accomplish steady-state operation of MPDTs. Here, general status of power sources in space is outlined.

For the power range from several tens of W to several kW, solar cells (< 10 kW, \( \alpha_{\text{pwr}} = 1-25 \text{ kg/kW} \)) or radio-thermal generators (< 1 kW, \( \alpha_{\text{pwr}} \approx 185 \text{ kg/kW} \)) are suitable. However, to obtain a high electric power of > 50 kW, development of a nuclear reactor will be necessary as shown in Fig. 1.2. Actually, there were some practical applications of nuclear reactors in space as shown in Table 1.4. From a historical point of view, the United States has flown only one reactor SNAP-10, and other reactors were launched by Russia. However, in terms of specific mass, the nuclear reactors are inferior to solar cells, because thermoelectric energy conversion had been adopted for the reactors. On the other hand, the recent project of SAFE-400 aims to reduce weight drastically by adopting a Brayton power system with heat pipe technology, which will result in a considerable low specific mass of about 5 kg/kW [45].

<table>
<thead>
<tr>
<th>Reactor</th>
<th>SNAP-10</th>
<th>SP-100</th>
<th>Topaz-1</th>
<th>Topaz-2</th>
<th>SAFE-400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Country</td>
<td>USA</td>
<td>USA</td>
<td>Russia</td>
<td>Russia-USA</td>
<td>USA</td>
</tr>
<tr>
<td>Thermal Output (kWt)</td>
<td>45.5</td>
<td>2000</td>
<td>150</td>
<td>135</td>
<td>400</td>
</tr>
<tr>
<td>Electrical Output (kWe)</td>
<td>0.65</td>
<td>100</td>
<td>5-10</td>
<td>6</td>
<td>100</td>
</tr>
<tr>
<td>Mass of Reactor (kg)</td>
<td>435</td>
<td>5422</td>
<td>&lt; 390</td>
<td>320</td>
<td>512</td>
</tr>
<tr>
<td>( \alpha_{\text{pwr}} ) (kg/kWe)</td>
<td>669</td>
<td>54</td>
<td>&lt; 78</td>
<td>53</td>
<td>5</td>
</tr>
</tbody>
</table>

1.2.7 Low-power Operations

The feature of availability with high power above 100 kW is valuable advantage for MPDTs. This fact, however, implies the difficulty to apply an MPDT to some practical application due to the absence of a suitable power source at the present stage. For this reason, operation with a lower power range (<100 kW) is also important. One of the candidates for such a low-power mode will be an operation with pulsed discharge. In this mode, time-averaged input power can be reduced to several kW level by reducing a duty ratio with holding a peak power. As a preliminary step, space experiments of
MPDTs with pulsed operation have been conducted as shown in Table 1.5.

Another approach for low-power operation is to apply an external magnetic field to a conventional SFMPDT, that is, AFMPDT. Application of a magnetic field provides some additional acceleration mechanism as mentioned in the Subsec. 1.2.5, thus discharge current for a particular thrust level can be reduced compared to SFMPDTs.

<table>
<thead>
<tr>
<th>Launch</th>
<th>Experiment</th>
<th>Propellant</th>
<th>Power (kW)</th>
<th>Sponsor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1975</td>
<td>Kosmos-728</td>
<td>Potassium</td>
<td>3</td>
<td>Russia</td>
</tr>
<tr>
<td>1977</td>
<td>K-9M-58 (suborbital)</td>
<td>Helium</td>
<td>0.02</td>
<td>ISAS</td>
</tr>
<tr>
<td>1980</td>
<td>MDT-2A (suborbital)</td>
<td>Ammonia</td>
<td>0.02</td>
<td>ISAS</td>
</tr>
<tr>
<td>1983</td>
<td>SEPAC/Spacelab-1</td>
<td>Argon</td>
<td>0.13</td>
<td>ISAS</td>
</tr>
<tr>
<td>1995</td>
<td>EPEX/SFU-1</td>
<td>Hydrazine</td>
<td>0.43</td>
<td>ISAS/IHI+MELCO</td>
</tr>
</tbody>
</table>

1.3 Motivations and Objectives

Since we have not obtained an optimal working condition, we need to explore various possibilities of MPDTs from a broad perspective. To predict flowfields and/or performances under a variety of conditions, development of a numerical code to simulate plasma flowfield in a MPDT is desired. For numerical studies, comparison of simulated flowfield and performance with experimental results is one of the most important matters, although it is generally difficult to observe overall flow structure within a thruster. To understand overall flow structure experimentally, a 2D-MPDT was developed, and precious two-dimensional images were captured [14,47,48]. Although these results were compared to the simulation results in Ref. [49], detailed quantitative arguments was not conducted, because the effects of viscosity, thermal conduction, ionization processes, thermal-nonequilibrium, and Hall effect were ignored in the previous calculation. In this thesis, a numerical code incorporating these physical effects is newly developed, and detailed comparison of simulated results with the two-dimensional images of a 2D-MPDT is conducted to understand plasma flowfields within a thruster, and simultaneously to validate calculated results.

Phenomena around a critical current are also focused on in this thesis. It is desired to understand the plasma behavior in a SFMPDT operated around a critical current $J_c$. This
Chapter 1 Introduction

study focuses on anode starvation, since specific discussion on current-carrier shortage with a detailed physical modeling has not been conducted as mentioned in Subsec. 1.2.4. In this study, plasma behavior around critical current is examined with a detailed physical modeling, and a method to suppress the current-carrier shortage is proposed.

Furthermore, this thesis attempts to examine low-power operations of MPDT with a pulsed SFMPDT, and an AFMPDT (steady-operation) respectively. Although performance measurements or system analyses for pulsed MPDTs have been conducted [50-54], numerical study on transient plasma behavior is limited to the work by Auweter-Kurtz et al. [55] in which discharge current was much lower than the critical current, and some important effects such as ionization processes, viscosity, and thermal conduction were ignored. In this study, temporal evolution of the plasma and performance operated in a pulsed mode at a critical current with detailed physical modeling are discussed. The effects of current profile on flowfields and performances are focused on, since a characteristic time of transient plasma behavior is generally in the same order as a characteristic time for current to rise. One of key issues for a pulsed operation will be how to enhance specific impulse. There is a concern that specific impulse of pulsed MPDT can be deteriorated by a time-lag of discharge ignition after starting propellant injection. In this subject, the effects of propellant species (argon, and helium respectively) on performances, especially specific impulse, are examined.

As for AFMPDT, owing to its fairly complicated acceleration mechanisms, the detailed acceleration processes and optimal working conditions are still under debate. Since some acceleration mechanisms coexist, clarification of the thrust components and their production mechanisms of AFMPDT will be of importance. Although some numerical studies have been conducted in the past, the numerical data are limited to some conditions, and thrust components and energy conversion processes within a thruster have not been clarified. In this study, the effects of strength and/or configurations of applied magnetic field on flowfields are examined, and thrust components and energy conversion processes are discussed. After that, remarks for performance improvement obtained from the arguments on the calculated results are addressed.

In this thesis, before starting the main subjects, physical aspects of interested flow and numerical modeling to analyze the flow are discussed in Chapter 2. The objectives
of the following chapters can be summarized as follows.

Chapter 3: Detailed understanding of plasma flowfields through comparisons of calculated results for a 2D-MPDT with experimental data are aimed.

Chapter 4: Plasma behavior in a coaxial SFMPDT operated around critical current, and suppression of anode starvation are discussed.

Chapter 5: Flowfields and performances with low-power operation modes of a pulsed SFMPDT and an AFMPDT are investigated.
In order to simulate flow properties and performances of MPDTs, comprehensive modeling of plasma flowfields is of considerable importance. In this chapter, physical aspects governing the plasma in MPDTs and numerical modeling to simulate the flowfield are discussed. At first, the characteristics of a plasma flow interested in this study are shown, and then assumptions and physical models appropriate to simulate the flow are explained. Actually, since physical models employed in each chapter are somewhat different, the common elements used in all themes are firstly extracted here. A numerical technique to solve the governing equations is also explained in this chapter.

2.1 Physical Backgrounds

When we construct a physical model for plasma flow under consideration, a lot of attention should be paid to selecting appropriate approximation to simulate actual phenomena of plasma. In this section, flow properties in an MPDT are described before showing governing equations.

2.1.1 Fluid Approximation

Plasma constituted from many particles (neutral, ion, electron) has microscopic and macroscopic properties. Particle-scaled microscopic dynamics of plasma should be dealt with a particle-in-cell (PIC) method, which can take into account almost all complex
phenomena occurring in plasma. However, when macroscopic property of plasma is interested in, this method is computationally too expensive despite the progressive computational technology. Since plasma in an MPDT operated under a typical working condition has an electron number density of $10^{20} – 10^{22} \text{ m}^{-3}$, a pressure of $0.1 – 10 \text{ Torr}$, and a temperature of $1 – 10 \text{ eV}$, PIC method will not be practical to simulate such plasma on the scale of several centimeters.

Provided that velocity distribution functions of each species reach Maxwellian in an adequately short time in comparison with the characteristic time of flowfield, the fluid approximation, which allows us to describe macroscopic properties of plasma, can be adopted to represent the governing physics. In terms of the fluid dynamics, validity of the fluid approximation is ensured by the condition that Knudsen number is less than 0.01:

$$K_n = \frac{\lambda_{ij}}{L_c} = \frac{1}{L_c n_i Q_{ij}} < 0.01 \, . \quad (2.1)$$

Since the plasma in an MPDT is believed to be fully ionized,* the charged particles are supposed to play a central role in attaining a Maxwellian distribution by collisions. Thus the momentum transfer collision cross section in Eq. (2.1) should be based on the Coulomb collision cross section [56]:

$$Q_{ij} = 5.85 \times 10^{-10} \ln \frac{A}{T_e^2}, \quad A = 1.24 \times 10^7 \sqrt{\frac{T_e^3}{n_e}} \, , \quad (s = e, i) \, . \quad (2.2)$$

When $T_{e,i} = 3 \text{ eV}$, $n_e = 1 \times 10^{21} \text{ m}^{-3}$, and $L_c = 1 \times 10^{-2} \text{ m}$, however, the estimated Knudsen number slightly exceeds 0.01 as shown in Table 2.1. Despite this fact, the preceding numerical studies showed that the fluid approximation of the plasma flow in MPDTs appears to provide reasonable results. One of the reasons for this may be that induced magnetic field perpendicular to a flow encourages the fluid approximation by playing the role of collisions in maintaining the Maxwell distribution function [57]. In the case of AFMPDT, the radial component of the applied magnetic field will offer the same effect. Another reason may be the existence of anomalous resistivity in actual flowfields, which enhances effective collision frequency via particle-wave interaction.

---

* The ionization fraction depends on the gas species. In the case of argon propellant, the ionization fraction is supposed to reach unity.
The discussion so far is limited to the parameters within a thruster. When it comes to a plume region outside the thruster, number densities of ions and electrons are decreased to $O(10^{20})$ m$^{-3}$ or less by aerodynamic expansion, which may lead to invalidity of the fluid approximation. However, taking account of a decrease in heavy-particle temperature due to aerodynamic expansion, the Knudsen number with regard to ion-ion collision remains in the order of $O(10^{-2})$. For example, when $T_i = 0.8$ eV and $n_i = 1 \times 10^{20}$ m$^{-3}$, the Knudsen number of ion-ion collision is estimated to be $1.6 \times 10^{-2}$. However, if $n_i = 1 \times 10^{19}$ m$^{-3}$, the Knudsen number increases up to $1.5 \times 10^{-1}$. Consequently, the fluid approximation for ions is considered to be valid unless the number density is decreased to less than $O(10^{19})$ m$^{-3}$, although this consideration depends on the temperature of ions. On the other hand, as for electrons, electron temperature maintains a relatively high value even in a plume region. When $T_e = 1.5$ eV and $n_e = 1 \times 10^{20}$ m$^{-3}$, the Knudsen number with regard to electron-electron collision is estimated to be $6.6 \times 10^{-2}$, therefore the fluid approximation for the electrons may be doubtful in a plume region. However, validity of the fluid approximation is expected to maintain by virtue of the presence of a magnetic field. For this reason, the fluid approximation is employed in this study.

### Table 2.1 Typical parameters of the flowfield in an MPDT (Argon is used for a heavy particle).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{ei}$ (eV)</td>
<td>3</td>
</tr>
<tr>
<td>$n_e$ (m$^{-3}$)</td>
<td>$1 \times 10^{21}$</td>
</tr>
<tr>
<td>$B$ (T)</td>
<td>0.1</td>
</tr>
<tr>
<td>$L_c$ (m)</td>
<td>0.01</td>
</tr>
<tr>
<td>Knudsen number (-)</td>
<td>$2.6 \times 10^{-2}$</td>
</tr>
<tr>
<td>Debye length (m)</td>
<td>$4.1 \times 10^{-2}$</td>
</tr>
<tr>
<td>Ion Larmor radius (m)</td>
<td>$1.6 \times 10^{-2}$</td>
</tr>
<tr>
<td>Electron Larmor radius (m)</td>
<td>$5.8 \times 10^{-2}$</td>
</tr>
<tr>
<td>Electron Hall parameter</td>
<td>4</td>
</tr>
<tr>
<td>Ion Hall parameter</td>
<td>$1.5 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

#### 2.1.2 Characteristics of the Plasma in an MPD Thruster

In the previous subsection, validity of the fluid approximation was discussed in view of the Knudsen number. Here, other important parameters relating with construction of
our physical model are focused on.

The characteristic length scale of charge separation is the Debye length:

$$\lambda_D = \sqrt{\frac{e_0 k T_e}{n_e e^2}} \approx 69.0 \frac{T_e}{n_e}.$$  \hspace{1cm} (2.3)

In view of the typical plasma properties in MPDTs, the Debye length is much smaller than the characteristic length as shown in Table 2.1, thus the plasma can be regarded as electrically neutral.

The Larmor radiuses of an ion \(r_{Li}\) and an electron \(r_{Le}\)

$$r_{Li} = \frac{u_{Li}}{\omega_{ci}}, \quad r_{Le} = \frac{u_{Le}}{\omega_{ce}} \quad \left( u_{Li} = \frac{2kT_i}{m_i}, \quad \omega_{ci} = \frac{eB}{m_i} \right),$$  \hspace{1cm} (2.4)

are also shown in Table 2.1 for the characteristic magnetic field of 0.1 T, which gives an measure of finite Larmor radius effects. In this case, it can be seen that the \(r_{Li}\) is in the same order with the characteristic length \(L_c\), whereas the \(r_{Le}\) is far less than \(L_c\). Although the large Larmor radius of ions may impede the continuum approach, not a little collision anticipated from low Knudsen number in the order of \(O(10^{-2})\) will moderate the finite Larmor radius effects.

It can be easily anticipated that low-pressure arc discharge (~ 0.1-10 Torr) in an MPDT leads to strong Hall effect characterized by Hall parameter

$$\beta_i = \frac{\omega_{ci}}{v_{ii}}, \quad \beta_e = \frac{\omega_{ce}}{v_{ei}} \quad \left( v_{jk} = n_k Q_{jk} \frac{8kT_j}{\pi m_j} \right),$$  \hspace{1cm} (2.5)

which are also tabulated in Table 2.1 under the plasma condition shown in the table. In Eq. (2.5) fully ionized plasma is assumed. The electron Hall parameter \(\beta_e\) of \(O(10^0)\) will result in distortion of current path within a discharge chamber, whereas the Hall effect with regard to an ion seems to be negligible. If number density of propellant is significantly decreased at some points, the effect of ion slip may have to be incorporated. However, since propellant is generally fully ionized within a thruster, the ion slip is not important [56].

Needless to say, we need to incorporate the effect of an induced magnetic field, which plays an essential role in plasma acceleration of SFMPDT. In a flow in SFMPDT, induced magnetic field can be transferred by convection, and/or can diffuse into plasma, which depends on magnetic Raynolds number \(R_m\):
\[ R_m = \mu_0 \sigma L U_c. \]  

Here, \( U_c \) denotes a characteristic velocity. Within a thruster, plasma velocity amounts to about \( 10^3 \)– \( 10^4 \) m/s, and intensive Joule heating by kA-level discharge current increases electron conductivity up to the order of \( 10^4 \) S/m, then the \( R_m \) can exceed unity.

### 2.2 Physical Modeling

On the basis of physical properties of a flow under consideration, construction of physical modeling can be completed. In this section, assumptions for modeling and fundamental equations governing a flowfield are described.

#### 2.2.1 Assumptions for Modeling

Since the conditions of ionization and thermal equilibrium are thought not to be satisfied as indicated from the experiment of Ref. [59], nonequilibrium ionization processes and thermal nonequilibrium are taken into account. Commonly employed assumptions or calculation conditions for all of the results of this thesis are summarized as follows.

- Two-dimensional flow structure is assumed. (axisymmetric or uniform in \( z \)-direction)
- Nonequilibrium ionization and recombination processes are incorporated.
- Two-temperature model with the effects of viscosity and thermal condition is used.
- Resistivity and Hall effect are taken into account in an induction equation. Ions slip is ignored.

The physical models employed in each chapter are somewhat different. The individual physical models for each subject are as follows.

**Two-dimensional Self-Field MPDT (Chapter 3)**
- Collisional-radiative model of Ar-II coupled with nonequilibrium singly ionization

**Coaxial Self-Field MPDT (Chapter 4)**
- Nonequilibrium multivalent ionization of Ar up to Ar-VI
2.2 Physical Modeling

Pulsed MPDT (Chapter 5, Sec. 5.1)
- Nonequilibrium singly ionization of Ar and He respectively (Not mixture gas)

Applied-Field MPDT (Chapter 5, Sec. 5.2)
- Nonequilibrium singly ionization of Ar
- Vector potential field
- Effect of electron pressure gradient in the generalized Ohm’s law
- Anomalous resistivity

The spectroscopic experimental evidence with a coaxial SFMPDT for Ar propellant indicates that the plasma ejected from a thruster exit retains high ionization fraction \( \alpha \equiv \frac{n_e}{n_h} \) attributed to doubly ionized species Ar-III as well as Ar-II [59]. The multivalent ionization processes are taken into account only in Chapter 4.

The nonequilibrium ionization models for Ar and He are addressed in the next subsection. The collisional-radiative model of Ar-II for 2D-MPDT is shown in Sec. 3.3, and models for the analyses of AFMPDT flowfields are shown in Subsec. 5.2.1.

2.2.2 Governing Equations

Fluid dynamics consists of a set of conservation equations with regard to the conserved quantities; mass, momentum, and energy. Additionally, these equations have to be coupled with the electric and magnetic fields to simulate discharge phenomena.

To incorporate the nonequilibrium reactions, continuity equations of ion species in consideration of ionization and recombination processes have to be solved. As for the equation of motion, total momentum has to be conserved in the presence of viscosity and the Lorentz force which can be represented with magnetic pressure as well. The two-temperature model entails two conservation equations with regard to the internal energy of heavy particles and electrons respectively. Additionally, an induction equation of magnetic field has to be coupled with above equations. Here, these equations are shown, and detailed explanations for some terms are subsequently attached.

Total mass density

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 .
\]  

(2.7)
Mass density of \( i \)-fold ion species

\[
\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}) = \dot{\rho}_i. \tag{2.8}
\]

Momentum

\[
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left( \rho \mathbf{u} \mathbf{u} + \left( p + \frac{B^2}{2\mu_0} \right) \mathbf{I} - \frac{B B}{\mu_0} \right) = \nabla \cdot \mathbf{f}. \tag{2.9}
\]

Internal energy of heavy particles (neutrals and ions)

\[
\frac{\partial U_h}{\partial t} + \nabla \cdot (U_h \mathbf{u}) = -p_h \nabla \cdot \mathbf{u} + \Phi + \nabla \cdot (\lambda_h \nabla T_h) + \delta E. \tag{2.10}
\]

Internal energy of electrons and ionization energy

\[
\frac{\partial}{\partial t} (U_e + U_i) + \nabla \cdot [(U_e + U_i) \mathbf{u}] = -p_e \nabla \cdot \mathbf{u} + \frac{j^2}{\sigma} + \nabla \cdot (\lambda_e \nabla T_e) + \frac{5k}{2e} j \cdot \nabla T_e - \delta E. \tag{2.11}
\]

Induction equation

\[
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\nabla \times \left[ \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B} + \frac{1}{\mu_0 en_e} (\nabla \times \mathbf{B}) \times \mathbf{B} \right]. \tag{2.12}
\]

To close the equations of system, the equation of state and the Ampère’s law are coupled with the equations above:

\[
p = n_h kT_h + n_e kT_e, \tag{2.13}
\]

\[
j = \frac{1}{\mu_0} \nabla \times \mathbf{B}. \tag{2.14}
\]

In the derivation of Eq. (2.12), the following generalized Ohm’s law is coupled with the Faraday’s law:

\[
j = \sigma \left( \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{en_e} j \times \mathbf{B} \right). \tag{2.15}
\]

From here, some terms and coefficients in the governing equations (2.7)-(2.12) are explained in detail.

**Reaction rate**

The right hand side (RHS) of Eq. (2.8) represents a reaction rate concerning forward and backward reactions by electron impact ionization and three-body recombination.
2.2 Physical Modeling

\[ A^{(i+1)+} + e^- \leftrightarrow A^i + e^- + e^- \]

The reaction rate taking account of multivalent ionization can be given as follows:

\[
\dot{\rho}_i = m_b \left( k_{f,i} n_{i+1} n_e - k_{b,i} n_i n_e^2 - k_{f,i+1} n_i n_e + k_{b,i+1} n_{i+1} n_e^2 \right).
\]

(2.16)

As a forward reaction rates \( k_{f,i} \), the Lotz formula is employed for both Ar and He [21,60 -62]:

\[
k_{f,i} = 6.7 \times 10^{-13} \sum_{k=1}^{N} a_{k,i} q_{k,i} \left[ \frac{1}{P_{k,i}/T_e} \int_{P_{k,i}/T_e}^{\infty} x^{-\frac{3}{2}} \left( 1 + \frac{P_{k,i}/T_e}{P_{k,i}/T_e + c_{k,i}} \right) \frac{e^{-y}}{y} dy \right].
\]

(2.17)

Here, \( N \) represents a number of subshells, and \( P_{k,i} \) is a binding energy of electrons in a \( k \)-th subshell of \( i \)-fold ion. \( q_{k,i} \) is a number of equivalent electrons in the \( k \)-th subshell of \( i \)-fold ion. The coefficients of \( a_{k,i} \), \( b_{k,i} \), and \( c_{k,i} \) denote individual constants given in Ref. [61]. The computed forward reaction rates for Ar (up to doubly ionization) and He (only singly ionization) are shown in Fig. 2.1. Recombination rates \( k_{b,i} \) are given from the forward reaction rates and the equilibrium constants \( K_i(T_e) \), in which electron temperature is used, since collisions with electrons are primarily processes which determine ionization fraction:

\[
k_{b,i} = \frac{n_i}{n_{i+1} n_e} \frac{k_{f,i}}{K_i(T_e)}
\]

(2.18)

\[
K_i(T_e) = \frac{g^{i+1}}{g^i} \left[ \frac{2\pi m_i k T_e}{h^2} \right]^{3/2} \exp \left( -\frac{eV_i}{k T_e} \right).
\]

(2.19)

Here, \( g^i \) denotes degeneracy factor of \( i \)-fold ion species.

In this study, the ionization processes only between ground states of ions are taken into account. However, as far as the reactions of Ar are concerned, it is inferred that the forward reaction rates, and hence the backward reaction rates are slightly different from those under consideration, because effects of ionization processes from subshells as well as the outermost shell are included in the Lotz formula of Eq. (2.17). In this study, it is assumed that ions produced by ionization from different shells are identified. This identification may lead to slight overestimation of the recombination rates toward a ground state. However, since plasma ejected from an MPDT retains high ionization fraction even at far downstream region [59], there will be little influence of this identification on the analyses of ionization and recombination processes.
Electrical Conductivity, Viscosity and Thermal Conductivity

The electrical conductivity is basically given by

\[ \sigma = \frac{e^2 n_e}{m_e v_{eh}}. \]  \hspace{1cm} (2.20)

In the simulation of AFMPDT flowfield, the effect of anomalous resistivity is incorporated as will be shown in Subsec. 5.2.1. The viscous stress tensor is defined by

\[ \bar{\tau}_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right). \]  \hspace{1cm} (2.21)

The coefficient of viscosity is computed with the following equation [43]

\[ \mu = \frac{1}{2} a' m_b \sum_{s \neq e} n_s l_s, \]  \hspace{1cm} (2.22)

where \( a' \) is a constant slightly less than unity. The dissipation function which corresponds to the heating rate via frictions between fluid particles is defined as follows:

\[ \Phi = \bar{\tau}_{ij} \frac{\partial u_i}{\partial x_j} = 2\mu \left( e_{ij} e_{ij} - \frac{1}{3} \left( \frac{\partial u_k}{\partial x_k} \right)^2 \right), \]  \hspace{1cm} (2.23)
where
\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \]

The thermal conductivities of heavy particles and electrons are calculated from the relations below [56], where the Prandtl number \( P_r \) is set to 2/3 in this study:
\[ \lambda_h = \frac{\mu c_p}{P_r}, \quad \lambda_e = \frac{2.4 k^2 n_{Te}}{1 + v_{ei}/v_{eh}\sqrt{2} m_e v_{eh}}. \] (2.24)

The thermal conduction with regard to diffusion of electron enthalpy is also included at the fourth term in the RHS of Eq. (2.11). Here, momentum transfer collision cross sections are required to evaluate collision frequencies. The Coulomb collision between charged particles can be given by [56]
\[ Q_{ij} = \frac{5.85 \times 10^{-10} Z_s Z_j}{T_i^2} \ln \left[ 1.24 \times 10^7 \left( \frac{T_e}{n_e} \right)^{1/2} \right], \] (2.25)

where \( Z_s \) denotes a valence of ions. The other cross sections for Ar are [63-65] (in MKSA unit)
\[ Q_{in} = 8 \times 10^{-19}, \] (2.26)
\[ Q_{en} = \frac{2.336 \times 10^{-14}}{C_e} \left( \frac{kT_e}{e} \right)^{1.609} \exp \left[ 0.0618 \left( \ln \frac{kT_e}{e} \right)^2 - 0.1171 \left( \ln \frac{kT_e}{e} \right)^3 \right], \] (2.27)
\[ Q_{nn} = 2.57 \times 10^{-19} \left( 1 + \frac{169.9}{T_h} \right), \] (2.28)

and those for He are [65]
\[ Q_{in} = 1.2 \times 10^{-19}, \] (2.29)
\[ Q_{en} = \frac{4}{3} \times 5.65 \times 10^{-20}, \] (2.30)
\[ Q_{nn} = 1.18 \times 10^{-19} \left( 1 + \frac{80.3}{T_h} \right). \] (2.31)

It has to be noted that, in Chapter 4, following models are used for the viscosity and thermal conductivity [63]:

\[ \nu = \nu_0 \left( \frac{T}{T_0} \right)^n, \] (2.32)
\[ \lambda = \lambda_0 \left( \frac{T}{T_0} \right)^m, \] (2.33)
\[
\mu = 0.45 \sqrt{\frac{\pi}{2}} \sqrt{\frac{2m_e kT_e}{T_h}} \sum_{i=0}^{n_e} \frac{n_i}{Q_{ei}^2} \left( \frac{T_e}{T_h} \right)^{\frac{2m_e}{m_h}} + \sum_{j=0}^{n_j} n_j Q_{ij}, \tag{2.32}
\]

\[
\lambda_h = \mu \kappa_v, \tag{2.33}
\]

\[
\lambda_e = 2.8 \sqrt{\frac{\pi}{2}} \frac{n_e k^2 T_e}{\sqrt{m_e kT_e} \left( n_e Q_{ee} + \sum_{i=0}^{n_i} n_i Q_{ei} \right)}, \tag{2.34}
\]

As for the Coulomb collision cross section, following Gvosdover cross section is adopted in Chapter 4, and \( Q_{en} \) is set to \( 4 \times 10^{-20} \) m²:

\[
Q_{ij} = \frac{\pi}{4} \left( \frac{Z_e Z_j e^2}{4\pi\varepsilon_0 T_e} \right) \ln \left( 1 + 144 \pi^2 \frac{(e_0 kT_e)^3}{n_e e^{\alpha}(\alpha + 1)} \right). \tag{2.35}
\]

### Internal Energy and Ionization Energy

The internal energies of heavy-particles and electrons are defined as

\[
U_s = \frac{3}{2} n_s kT_s \quad (s = h, e). \tag{2.36}
\]

The total ionization energy in Eq. (2.11) is generally given by the sum of ionization energy the criterion of which is the ground state of the atom:

\[
U_i = \sum_{k=1}^{N} \sum_{j=1}^{k} V_j n_k. \tag{2.37}
\]

Here, \( N \) denotes the number of ion species under consideration. In Eq. (2.11), it is assumed that Joule heating primarily affects the internal energy of electrons and the ionization energy. Internal energy of electrons is transferred to heavy particles via collisions, the effect of which is represented by \( \delta E \) in Eqs. (2.10) and (2.11) [56]:

\[
\delta E = 3n_e \frac{m_e}{m_h} \nu_{eh} k(T_e - T_h). \tag{2.38}
\]

### Induction Equation and Current Path

In the formulation of a coaxial MPDT, it is useful to convert an azimuthal magnetic field component into a stream function of current:

\[
\psi \equiv rB_\theta. \tag{2.39}
\]
With this stream function, current density can be given in the following forms:

\[ j_z = \frac{1}{\mu_0 r} \frac{\partial \psi}{\partial r}, \quad j_r = -\frac{1}{\mu_0 r} \frac{\partial \psi}{\partial z}. \] (2.40)

When a cathode radius \( r_c \) and an induced magnetic field at a cathode root \((-\mu_0 J / 2\pi r_c)\) are taken as a characteristic length \( L_c \) and a characteristic magnetic field \( B_c \) respectively, dimensionless stream function \( \hat{\psi} \) at a certain point corresponds to a ratio of current flowing downstream viewed from the point under consideration \( (J^*) \) to the total discharge current \( J \):

\[ \hat{\psi} = \frac{r B_0}{-r_c \mu_0 J} = \frac{r \mu_0 J^*}{-r_c \mu_0 J} = \frac{J^*}{J}. \] (2.41)

In the case of 2D-MPDT, induced magnetic field itself plays the same role.

### 2.3 Numerical Procedures

The governing equations shown in the previous section are to be solved with the aid of Computational Fluid Dynamics (CFD). In this section, a typical computational region and boundary conditions are firstly shown, and then numerical techniques to obtain a solution are described. It has to be noted that electrodes geometry and computational regions for each subject are somewhat different, and input parameters as well. At the beginning of each subject, the geometries and the actual input parameters used for the analyses are specifically explained. Regarding the boundary conditions, the essence required to the numerical procedure is extracted in this section.

#### 2.3.1 Computational Region and Initial Condition

A domain under consideration has to be spatially discretized to solve a flowfield numerically. To recognize the image of calculations in this study, a computational region is firstly illustrated. The assumption of two-dimensional flow structure allows us to limit a computational region only in an upper half of a cross section of a flow. A typical computational region is described in Fig. 2.2. Since some fractions of discharge
current can be expanded toward the outside of a thruster, it is preferable to expand a computational region to a downstream plume region to capture entire aspects of a flowfield.

Before starting an iteration of time-marching, some sorts of initial conditions have to be set for the flowfield. For a steady-state solution, one does not have to be concerned about an initial condition seriously, thus a high-temperature steady flow with a constant ionization fraction is assumed for gasdynamic quantities. Aside from the gasdynamic quantities, initial magnetic field distribution determining an initial current path has also to be set. Then, with the initial gasdynamic quantities, the induction equation in a steady form is solved by an iterative method. On the other hand, for an analysis of a pulsed MPDT, initial temperature has to be low. Thus, a steady flow injected at 300 K is used as an initial condition.

![Fig. 2.2 Typical computational region for axisymmetric flow.](image)

2.3.2 Boundary Conditions

Since the partial differential equations shown in Subsec. 2.2.2 are hyperbolic, solutions must be computed by a time-marching method from initial data with appropriate boundary conditions for the computational region in Fig. 2.2.

**Gasdynamic Conditions**

At the inlet, a mass flow rate and temperatures of heavy particles and electrons are kept constant. Since an injection port of a propellant is located at $z = 0$ mm, propellant is
supposed to be heated immediately at the thruster inlet by the Joule heating. Thus, to circumvent the difficulty of considering plasma ignition problem, it is assumed that high temperature plasma is injected from the inlet. Then, relatively low ionization fraction $\alpha$ compared to the bulk plasma is assumed at the inlet. Regarding other physical quantities of the fluid, inlet conditions are switched in accordance with whether the flow is supersonic or subsonic at the inlet. When the flow is supersonic at the inlet, the streamwise velocity is set to an acoustic velocity, and then the mass density is determined from a given mass flow rate. The pressure can be obtained from the equation of state. In the case of subsonic, the pressure is given by first-order extrapolation as backward information, and then the mass density is determined from the equation of state. The inlet velocity can be given from a given mass flow rate.

A non-slip condition is imposed on the velocity along the walls. The heavy particle temperature $T_h$ on each electrode is limited by the temperature in consideration of a melting point of electrode materials. In this study, $T_h$ is kept to be less than 1,300 K on the anode, and to be less than 2,000 K on the cathode, where the anode and the cathode is assumed to be copper and Th-loaded tungsten respectively. Regarding the electron temperature, it is assumed to be adiabatic or the condition of $T_e < 2$ eV is imposed on the walls.

As for outflow conditions, since a propellant is to be accelerated to a supersonic flow, all of the characteristic lines are basically directed downstream at the boundary of the plume region. Thus, all of the physical quantities are determined by zeroth extrapolation from upstream. However, when the outflow is subsonic, the pressure is set to an empirically reasonable value for a subsonic area.

**Conditions for Electromagnetic Field**

Aside from the gasdynamic quantities, magnetic field has to be prescribed at the boundaries to integrate the induction equation (2.12). Since the azimuthal component of the magnetic field $B_\theta$ is induced by discharge current, this component is to reflect a discharge current value. The $B_\theta$ produced by the discharge current $J$ at the inlet of a coaxial thruster can be determined from the Ampère’s law:

$$B_\theta = -\frac{\mu_0 J}{2\pi r}.$$  \hspace{1cm} (2.42)
The $B_z$ at the inlet of a 2D-MPDT with a width of $W$ can be given by the following equation under the assumption that the cathode is an infinite flat plate:

\[ B_z = -\frac{\mu_0 J}{2W}. \]  \hspace{1cm} (2.43)

The surface of the electrodes is assumed to be equipotential, i.e. the tangential component of electric field along the electrodes vanishes, which is reduced to the following relation in a dimensionless form for analyses of coaxial coordinate system.

\[ \frac{\partial \psi}{\partial \eta} \left[ \frac{\beta}{R_a} (\xi, \eta, + \xi, \eta, \xi, \eta, - \xi, \eta, \xi, \eta) \right] + \frac{\partial \psi}{\partial \xi} \left[ \frac{\beta}{R_a} (\xi, \eta, + \xi, \eta, \xi, \eta, \xi, \eta) \right] = 0. \]  \hspace{1cm} (2.44)

This provides the boundary condition for $B_\theta$ on the electrodes. On the insulator shown in Fig. 2.2, induced magnetic flux density is set to zero.

### 2.3.3 Numerical Schemes

The MHD equations (2.7)-(2.12) can be described in conservative form based on a finite volume method. For the axisymmetric coordinate system, the system equations can be represented as

\[ \frac{\partial Q}{\partial t} + \frac{\partial F}{\partial z} + \frac{\partial G}{\partial r} = S, \]  \hspace{1cm} (2.45)

where $Q$ denotes the conservative variables and $F$, $G$ are the physical flux related with the convective terms. The vector $S$ represents all of the source terms including subsidiary terms resulting from the axisymmetric coordinate system. Actually, for the analyses of coaxial MPDTs, the system equations multiplied by the volume of the cell $rdrd\theta dz$ are used to enhance conservation property (see Appendix A), but numerical procedure for Eq. (2.45) is described here for simplicity. The physical fluxes $F$, $G$ are represented with numerical fluxes by which a variety of numerical schemes are classified. The conservative form of Eq. (2.45) is to be converted to the generalized coordinate system $(\xi(z,r), \eta(z,r))$ in which actual calculation is performed:

\[ \frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{F}}{\partial \xi} + \frac{\partial \tilde{G}}{\partial \eta} = \tilde{S}, \]  \hspace{1cm} (2.46)

where

\[ \tilde{Q} = \frac{Q}{J}, \quad \tilde{F} = \frac{1}{J}(\xi, F + \xi, G), \quad \tilde{G} = \frac{1}{J}(\eta, F + \eta, G), \quad \tilde{S} = \frac{S}{J}. \]  \hspace{1cm} (2.47)
2.3 Numerical Procedures

Here, $J$ denotes a Jacobian with regard to the generalized transformation:

$$J = \frac{\partial (\xi, \eta)}{\partial (z, r)} = \begin{vmatrix} \xi_z & \xi_r \\ \eta_z & \eta_r \end{vmatrix}.$$  \hspace{1cm} (2.48)

There are some numerical solvers for MHD equation such as approximate Riemann solver [66] and HLL-based schemes [67]. In this study, the TVD Lax-Friedrich scheme [68,69] is adopted, because this scheme can be easily applied to an MHD solver without evaluating a Jacobian of physical fluxes, which offers a tremendous savings in computational cost. Additionally this scheme is relatively robust due to its intrinsic large numerical dissipation, feature of which is supposed to be suitable for the present purpose to examine the critical operation range of a MPDT. The spatial accuracy of second order is maintained by using MUSCL (monotonic upstream schemes for conservation laws) approach which evaluates a physical value on a cell surface by extrapolation within certain accuracy. Also, a predictor-corrector method is employed as a time integration technique to maintain second order accuracy for time, which has importance especially for the analysis of a pulsed MPDT. The predictor for a time step of $n+1/2$ can be given as follows:

$$\tilde{Q}_{i,j}^{n+1/2} = \tilde{Q}_{i,j}^n - \frac{\Delta t}{2\Delta \xi} (\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j}) - \frac{\Delta t}{2\Delta \eta} (\tilde{G}_{i,j+1/2} - \tilde{G}_{i,j-1/2}) + \frac{\Delta t}{2} \tilde{S}^n, \hspace{1cm} (2.49)$$

where

$$\tilde{F}_{i+1/2,j} = \left( \frac{\xi_z}{J} \right)_{i+1/2,j} \tilde{F}(\tilde{Q}_{i+1/2,j}) + \left( \frac{\xi_r}{J} \right)_{i+1/2,j} \tilde{G}(\tilde{Q}_{i+1/2,j}),$$

$$\tilde{G}_{i,j+1/2} = \left( \frac{\eta_z}{J} \right)_{i,j+1/2} \tilde{F}(\tilde{Q}_{i,j+1/2}) + \left( \frac{\eta_r}{J} \right)_{i,j+1/2} \tilde{G}(\tilde{Q}_{i,j+1/2}),$$

with the definitions of

$$\tilde{Q}_{i+1/2,j} = Q_{i,j}^n \pm \frac{1}{2} \Delta \tilde{Q}, \hspace{1cm} \tilde{Q}_{i,j+1/2} = Q_{i,j}^n \pm \frac{1}{2} \Delta \tilde{Q}.$$

The limited slopes of the conservative variables $\Delta \tilde{Q}, \Delta \tilde{Q}$ are evaluated with the minmod limiter, which impose the TVD property for the scheme:
\[ \Delta \overline{Q}_i = \minmod(\Delta Q_{i+1/2,j}, \Delta Q_{i-1/2,j}), \quad \Delta \overline{Q}_j = \minmod(\Delta Q_{i+1,j}, \Delta Q_{i,j+1}), \]

\[ \Delta Q_{i+1/2,j} = Q^e_{i+1,j} - Q^e_{i,j}, \quad \Delta Q_{i,j+1/2} = Q^e_{i,j+1} - Q^e_{i,j}, \]

where

\[ \minmod(x, y) = \sgn(x) \max[0, \min(|x|, \sgn(x) y)]. \]

The corrector for a time step of \( n+1 \) is evaluated with the quantities at \( n+1/2 \) obtained from Eq. (2.49):

\[ \tilde{Q}^{n+1}_{i,j} = \tilde{Q}^e_{i,j} - \frac{\Delta t}{\Delta \xi} (\tilde{F}^{n+1}_{i+1/2,j} - \tilde{F}^{n+1}_{i-1/2,j}) - \frac{\Delta t}{\Delta \eta} (\tilde{G}^{n+1}_{i,j+1/2} - \tilde{G}^{n+1}_{i,j-1/2}) + \Delta \tilde{S}^{n+1/2}, \quad (2.50) \]

\[ \tilde{F}^{n+1}_{i+1/2,j} = \frac{1}{2} \left[ \left( \frac{\xi}{J} \right)_{i+1/2,j} \left( F(Q^L_{i+1/2,j}) + F(Q^R_{i+1/2,j}) \right) \right. \]

\[ + \left( \frac{\xi}{J} \right)_{i+1/2,j} \left( G(Q^L_{i+1/2,j}) + G(Q^R_{i+1/2,j}) \right) \left( \Phi_{i+1/2,j} \right)_{j+1/2} \]

\[ \tilde{G}^{n+1}_{i,j+1/2} = \frac{1}{2} \left[ \left( \frac{\eta}{J} \right)_{j+1/2} \left( F(Q^L_{i,j+1/2}) + F(Q^R_{i,j+1/2}) \right) \right. \]

\[ + \left( \frac{\eta}{J} \right)_{j+1/2} \left( G(Q^L_{i,j+1/2}) + G(Q^R_{i,j+1/2}) \right) \left( \Phi_{i,j+1/2} \right)_{j+1/2} \]

\[ \frac{1}{J_{i+1/2,j}} = \frac{1}{2} \left( \frac{1}{J_{i,j}} + \frac{1}{J_{i+1,j}} \right), \quad \frac{1}{J_{i,j+1/2}} = \frac{1}{2} \left( \frac{1}{J_{i,j}} + \frac{1}{J_{i,j+1}} \right). \]

Here, the variables \( Q^L \), \( Q^R \) represents the extrapolated values at a cell surface via MUSCL approach*:

\[ Q^L_{i+1/2,j} = Q^e_{i+1/2,j} + \frac{1}{2} \Delta \overline{Q}_i, \quad Q^L_{i,j+1/2} = Q^e_{i,j+1/2} - \frac{1}{2} \Delta \overline{Q}_j, \]

\[ Q^R_{i+1/2,j} = Q^e_{i,j+1/2} + \frac{1}{2} \Delta \overline{Q}_i, \quad Q^R_{i,j+1/2} = Q^e_{i,j+1/2} - \frac{1}{2} \Delta \overline{Q}_j. \]

\( \Phi_{i+1/2,j} \) and \( \Phi_{i,j+1/2} \) in Eq. (2.50) show the numerical dissipation defined as follows:

\[ \Phi_{i+1/2,j} = -\max_{i+1/2,j} \left( Q^R_{i+1/2,j} - Q^L_{i+1/2,j} \right), \quad \Phi_{i,j+1/2} = -\max_{i,j+1/2} \left( Q^R_{i,j+1/2} - Q^L_{i,j+1/2} \right). \quad (2.51) \]

Since the difference between the left and right extrapolation is \( O(\Delta x^2) \), the diffusive effect is greatly reduced compared to the original first order Lax-Friedrich scheme.

* There are options in applying the limiters, i.e. one can impose limiters on the conservative, primitive, or characteristic variables. In this study, the limiters are imposed on the conservative variables [69].
\( c_{\text{max}} \) in Eq. (2.51) is a maximum propagation velocity of MHD waves in \((\xi, \eta)\) space. This can be determined based on the following consideration. In ideal MHD, three different waves exist; slow, Alfvén, and fast waves [66]. In the direction of \( n \), the following relationship generally holds:

\[
\frac{c_{\text{slow}}}{n_{\text{Alfven}}} \leq \frac{c_{\text{Alfven}}}{n_{\text{Alfven}}} \leq \frac{c_{\text{fast}}}{n_{\text{Alfven}}}, \quad (2.52)
\]

where

\[
c_{\text{Alfven}} = \frac{|B|}{\sqrt{\mu_0 \rho}}, \quad (2.53)
\]

\[
c_{n_{\text{fast,slow}}} = \frac{1}{2} \left[ c_{\text{sound}} + c_{\text{Alfven}} \pm \sqrt{\left( c_{\text{sound}} + c_{\text{Alfven}} \right)^2 - \left( 2c_{\text{sound}}c_{\text{Alfven}} \right)^2} \right]^{1/2}, \quad (2.54)
\]

with the definitions of

\[
c_{\text{sound}} = \sqrt{\frac{\gamma P}{\rho}}, \quad c_{\text{Alfven}} = \frac{|B|}{\sqrt{\mu_0 \rho}}.
\]

The plus sign in Eq. (2.54) corresponds to the fast magnetosonic wave speed. The relationship of Eq. (2.52) indicates that the largest wave speed by which information can propagate in the direction of \( n \) is

\[
|u_n| + c_{n_{\text{fast}}}. \quad (2.55)
\]

Therefore, the \( c_{\text{max}} \) in Eq. (2.51) can be defined as follows:

\[
c_{i,j+1/2} = \max \left( c_{i,j+1/2,1}^\text{max} (Q^R), c_{i,j+1/2,2}^\text{max} (Q^L) \right), \quad c_{i,j+1/2}^\text{max} = \max \left( c_{i,j+1/2,1}^\text{max} (Q^R), c_{i,j+1/2,2}^\text{max} (Q^L) \right),
\]

where

\[
\hat{c}_{\text{max}} = \left| \xi_z u_z + \xi_x u_x \right| + \xi_x c_{\text{fast}} + \xi_z c_{\text{fast}}.
\]

The conservative quantities can be time-marched with Eqs. (2.49) and (2.50), where central difference is used for the derivatives appearing in the source term \( S \).

### 2.3.4 Time Interval

A time interval for the time-marching method can be estimated based on some different characteristic time scales. In the case of ideal MHD, one has only to take into account the time scale of convection:

\[
\Delta t_{\text{conv}} \leq \frac{\Delta x}{|u| + c_{n_{\text{fast}}}}. \quad (2.56)
\]
On the other hand, the resistive and the Hall MHD equation with dissipative effects of the viscosity and the thermal conduction requires much smaller characteristic time scales. Even though the convection of propellant is of primary importance to propulsion, the characteristic time scale of the dissipative effects is normally shorter than $\Delta t_{\text{conv}}$, because it is proportional to $\Delta x^2$. Among the dissipative phenomena, one of the severest phenomena is the magnetic diffusion, the time scale of which is

$$\Delta t_{\text{magdiff}} \leq \mu_0 \sigma \Delta x^2 = O\left(10^{-10} - 10^{-11}\right) \text{ s.} \quad (2.57)$$

Basically the time interval of every step is determined to satisfy this restriction. However, the characteristic time concerning the Hall term can be severer than $\Delta t_{\text{magdiff}}$ [70]. In such a case, the following characteristic velocities have to be taken into account in Eq. (2.56) in addition to the fast wave:

- Hall velocity: $V_{H} = -|\mathbf{j}|/e n_e$, 
- Hall drift wave: $V_{HDW} = c_{\text{Alfven}}^2 / L_n \omega_{ci}$, 
- Whistler wave: $V_w = k c_{\text{Alfven}}^2 \omega_{ci}$.

Here $L_n = (\partial \ln n_e / \partial x)^{-1}$ is the density gradient scale length.

For a steady state solution, local time step technique, by which a time interval is locally determined for each cell, is employed under the restrictions above to accelerate the convergence [71], although calculations of a pulsed MPDT are conducted with a global time step. However, since the restriction of Eq. (2.57) is actually quite severe in a calculation of a pulsed MPDT, which needs to start the time-marching from a cold flow, the magnetic diffusion term is treated with an implicit method for the analysis of pulsed MPDT flowfields [71, 72].
Chapter 3.

Fundamental Properties of Plasma Flowfields in an MPD Thruster

A two-dimensional MPD thruster enables us to observe plasma flowfield directly via some spectroscopic techniques, and to easily obtain two-dimensional image data, whereas the same measurements are hard for a coaxial thruster. Experimental data on overall flow structure inside a thruster help us to understand physical processes, and to validate numerical results. In this chapter, detailed comparisons of calculated results with measured data within a 2D-MPDT are conducted. Through comparisons of the experimental data of current path, electron number density, and electron temperature with calculated results, detailed understandings on fundamental plasma properties in an MPDT and validation of calculated results are aimed.

3.1 Thruster Configuration

The thruster adopted here consists of a flared anode and a short cathode with a conical tip as shown in Fig. 3.1. In accordance with the geometry used in the experiments [14], the inner interval between the anodes is set to 28 mm at the inlet, and 56 mm at the outlet of the thruster. The cathode has a thickness of 8 mm. The width of the thruster $W$ is assumed as a constant of 80 mm. The computational region is limited to upper half domain, and is extended toward the downstream domain up to 100 mm in the streamwise direction.
Chapter 3 Fundamental Properties of Plasma Flowfields in an MPD Thruster

3.2 Calculation Conditions

The calculation conditions are set in accordance with the preceding experimental conditions. Argon is used as a propellant, and the prescribed mass flow rate of 1.25 g/s is set at the inlet. The plasma at the inlet is assumed to have the heavy particle temperature of 5000 K and the electron temperature of $10^4$ K in this calculation. The ionization fraction at the inlet is assumed to be 0.1. The total discharge current $J$ is varied over the range of 8 - 12 kA. These conditions are summarized in Table 3.1.

<table>
<thead>
<tr>
<th>Propellent</th>
<th>Ar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass flow rate (g/s)</td>
<td>1.25</td>
</tr>
<tr>
<td>Discharge current (kA)</td>
<td>8, 10, 12</td>
</tr>
<tr>
<td>Inlet heavy particle temperature (K)</td>
<td>$5\times10^3$</td>
</tr>
<tr>
<td>Inlet electron temperature (K)</td>
<td>$1\times10^4$</td>
</tr>
<tr>
<td>Inlet ionization fraction</td>
<td>0.1</td>
</tr>
<tr>
<td>Electron temperature at walls</td>
<td>adiabatic</td>
</tr>
</tbody>
</table>

3.3 Collisional-Radiative Model of Ar-II

The model used in the analysis of 2D-MPDT flowfields includes collisional-radiative (CR) processes of Ar-II, which has not been taken into account in the other numerical studies. In the comparison of the calculated electron temperature with the measured data, inclusion of the CR model for Ar-II is expected to provide some insights, since the measured electron temperature was evaluated with the relative intensity of Ar-II lines.
With this model, we can examine the validity of assumption of local thermodynamic equilibrium in the flow. Moreover, direct comparison of the distribution of the excited ions with the radiative intensity observed in the experiment becomes possible.

Since it was reported that, in the experiment, the intensity of the spectral lines of Ar-I and Ar-III were far less than that of Ar-II [14], the excitation of Ar-I and the ionization from Ar-II to Ar-III are not taken into account here. The CR model for Ar-II employed here deals with the interactions among the doublet ions, where some degenerate levels are modeled as a homogeneous group, which was originally established for the simulation of plasma properties in an argon ion laser by Pots [73]. Since ion and electron temperatures and plasma density in an argon ion laser ($T_i \approx 1 \text{ eV}, T_e \approx 4 \text{ eV}, n_e \approx 10^{20} \text{ m}^{-3}$) are close to those of the flowfields under consideration, the CR model is considered to be adapted to the present simulation. In the CR model [73], collisional coupling between quartet and doublet levels were ignored due to lack of data, although the requirement to include possible interactions between quartet and doublet systems was recognized in the literature. Generally, excitation cross sections corresponding to optically forbidden transitions can have maximum values of the same order as those for optically allowed transitions, but they tend to decrease more rapidly with increasing electron energy [56]. Therefore, large number of quartet ions will be certainly produced in a thruster, since typical electron temperature is far less than the excitation energy from the ground state. Thus, absolute values of the excited doublet ions derived from the simulation with this model will not be correct. However, relative values of the populations in each level, which play an important role on evaluating the excitation temperature discussed later, are considered to be reasonable, because strong couplings between quartet and doublet ions in the upper levels were not observed in the perturbation spectroscopy of an argon laser discharge [74].

The original model in Ref. [73] takes into account excited states up to 5g, but the uppermost energy level is limited to 4p orbit for simplicity as shown in Fig. 3.2 in this study. Thus the states for Ar-II of 3p (ground), 4s, 3d, and 4p are involved, where the 3d level is subdivided into three groups ($^2\text{P}$, $^2\text{F}_{5/2}$, $^2\text{F}_{7/2}$, and $^2\text{D}$). Regarding the excitation rate, the formula given by Rubin and Sobolev [75] is used for all reactions except for 3p-3d ($^2\text{P}$, $^2\text{F}_{5/2}$), 3p-3d ($^2\text{D}$), 3d ($^2\text{P}$, $^2\text{F}_{5/2}$)-4p, and 3d ($^2\text{D}$)-4p for which Beigman’s data are utilized [76].
Fig. 3.2 Collisional-Radiative model of Ar-II [73]. The energy levels shown are measured from the ground state of Ar-II.

The continuity equations for excited ions are to be solved apart from Eqs. (2.7) and (2.8) in consideration of all relevant reactions. Also, the internal electron energy and the ionization energy of Eq. (2.11) have to be slightly altered taking account of the excitation energy and the radiation energy $q_{rad}$. In this study, the plasma is assumed to be optically thick towards the ground state, and to be optically thin for the inter-level transfers [77]:

$$\frac{\partial}{\partial t} \left( U_e + U_{i,ex} \right) + \nabla \cdot \left[ \left( U_e + U_{i,ex} \right) \mathbf{u} \right] = -p_e \nabla \cdot \mathbf{u} + \frac{j^2}{\sigma} + \nabla \cdot \left( \lambda_e \nabla T_e \right) + \frac{5k}{2e} j \cdot \nabla T_e - \delta E - q_{rad}, \quad (3.1)$$

where

$$U_{i,ex} = V_i n_i + \sum_{ex} (V_i + V_{ex}) n_{ex}, \quad (3.2)$$

$$q_{rad} = \sum_{i>j} eV_{ij} A_{ij} n_j. \quad (3.3)$$

Here, values of $V_i$ and $n_i$ are the ionization energy for Ar-I, and the number density of the ions in the ground state respectively. The $V_{ij}$ and $A_{ij}$ denote the energy gap $(V_i - V_j)$ between excited levels and the Einstein transition probabilities, for which the values found in Ref. [73] are used. The summations for Eqs. (3.2) and (3.3) are taken for all excited states under consideration, and for all relevant radiations respectively.
3.4 Comparisons of Flowfields

Comparisons with the experimental data of 2D-MPDT obtained by Toki [14], Nakayama [47], and Funaki [48] enable us to validate the numerical results. As for the measured data within the thruster, the current path, electron number density, and electron temperature are available so far. In the experiments, the current path was measured with a magnetic-sensitive film inserted into the thruster which reveals the magnetic field strength from darkness patterns. The electron number density was measured by the Mach-Zehnder interferometry [47,48], or was obtained from the absolute intensity method [14]. Regarding the electron temperature distribution, a relative intensity method of spectroscopy was employed [47].

3.4.1 Current Path

The numerical and the experimental results of the current path for $J = 12$ kA are shown in Fig. 3.3. The labeled values on the contour lines denote the ratio of the current flowing upstream viewed from the line to the total discharge current. The calculated result shows that, in the flared region, the current path is obliquely skewed due to the Hall effect [78], where the highest Hall parameter amounts to about 40 in the vicinity of the anode surface. The increase in the Hall parameter is attributed to depletion of the plasma density due to expansion in the flared nozzle. Although this skewed current path, especially near the anode surface, may not appear significantly in the experimental results, it is appropriate to suppose that both the numerical and experimental results suggest about 20-30% of the discharge current concentrates at the anode edge. Without incorporating the Hall effect, the current concentration at the anode edge cannot be obtained as indicated in [49]. The difference in current contour in the vicinity of the flared anode surface may be attributed to neglecting the sheath effects in the modeling, and/or to the effect of anomalous resistivity which will moderate the Hall parameter.

3.4.2 Electron Number Density

The numerical and the experimental results of the electron number density $n_e$ for $J = 12$ kA are shown in Fig. 3.4. Although there seems to be some quantitative disagreement between the experimental results of Fig. 3.4-(b) [47] and Fig. 3.4-(c) [48],
these results indicate that the electron number density is in the order of $10^{20} - 10^{21}$ m$^{-3}$ within the thruster. Also, the result of the absolute intensity method indicates $n_e$ is in the order of $10^{21}$ m$^{-3}$ [14]. The two-dimensional measured images of $n_e$ show that almost all propellants are ionized in the vicinity of the inlet part, and $n_e$ has a maximum basically around the cathode.

The numerical result shows $n_e$ amounts to the maximum about $4 \times 10^{21}$ m$^{-3}$ around the cathode, which is between the two different experimental data. The high density at the cathode tip, which did not appear specifically in the numerical result of Ref. [49], is attributed to the radial pinch force enhanced by the Hall effect leading to the obliquely skewed current profile. In the vicinity of the flared anode surface, $n_e$ is considerably decreased in comparison with the other region due to the pinch force and/or expansion through the flared nozzle, which cannot be definitely recognized from the experimental results due to their resolution near the anode. It can be seen that the simulated result predicts an existence of a compression wave extended from the cathode tip where the plasma is compressed by the pinch force. However, the compression wave cannot be apparently seen in the measured distribution obtained by means of Mach-Zehnder interferometry, whereas a fan-shaped distribution, the center of which is the cathode as seen in the experimental result of Fig. 3.4-(b), is well captured by the calculation. Since a hemispherical cathode tip employed in the experiment is modeled by the conical cathode tip for simplicity, the compression wave may be accentuated in the calculation.

Regarding ionization of the propellant, the ionization fraction amounts to unity near the inlet part as predicted from the experiments.
3.4 Comparisons of Flowfields

Fig. 3.3 Current path, 2D-MPDT, Ar, 1.25 g/s, \( J = 12 \) kA; (a) Calc. (b) Exp. [14], (c) Exp. [48].
Fig. 3.4 Electron number density (m$^{-3}$), 2D-MPDT, Ar, 1.25 g/s, $J = 12$ kA; (a) Calc., (b) Exp. [47], (c) Exp. [48].
3.4.3 Electron Temperature

In the previous numerical study [49], since fully ionized and thermal equilibrium ($T_h = T_e$) plasma flow was assumed, quantitative comparison of the temperature was circumvented. Since the two-temperature model is incorporated in the present model, electron temperature distribution can be discussed more closely. In Fig. 3.5, the numerical and the experimental results of the electron temperature $T_e$ are described. According to the experimental result, $T_e$ amounts to about 2.5 eV at the cathode tip and around the inlet part, while it is no more than 1.2 eV near the anode surface. On the other hand, $T_e$ of the numerical result ranges from about 3 eV around the cathode tip to about 4 eV near the anode surface. The reason for the high electron temperature near the anode is attributed to the depletion of the plasma density as shown in Fig. 3.4-(a). While the measured $T_e$ near the anode is less than that near the cathode, the calculated $T_e$ near the anode is higher than that near the cathode. This discrepancy needs careful discussion.

Different from a coaxial MPDT, in 2D-MPDT, the current density near the anode is not much lower than that near the cathode because of its configuration. Thus if the plasma density near the anode is lower than that near the cathode, the Joule heating per unit mass near the anode can become higher than that near the cathode, which can result in higher electron temperature near the anode. From this standpoint, the calculated result seems reasonable. However, the current density near the cathode is actually higher than that near the anode due to the small cathode size compared with the anode, thus the Joule heating per unit volume around the cathode can be higher than that near the anode. From this standpoint, the tendency observed in the experiment seems acceptable.

In order to explain this discrepancy, the method of the electron temperature measurement is focused on. The two-dimensional image of $T_e$ given in Ref. [47] was obtained with a relative intensity method, where the intensity of spectral lines of Ar-II was utilized, and then the electron temperature was estimated with the assumption of partial local thermodynamic equilibrium (LTE), i.e. the measured temperature is to be interpreted as excitation temperature. Therefore, if the distributions of the excited ions of Ar-II associated with the utilized spectral lines can be computed with the CR model for Ar-II shown in Sec. 3.3, it becomes possible to evaluate the excitation temperature.
$T_{ex}$ by assuming LTE relation for the obtained populations of the excited ions. This calculated $T_{ex}$ is to be compared to the measured $T_e$. According to the wavelength found in Ref. [47], the radiated light with regard to the transition from 4p to 4s orbit was mainly used for the measurement, hence the excitation temperature $T_{ex}$ can be defined by the following equation:

$$\frac{n_{4p}}{n_{4s}} = \frac{g_{4p}}{g_{4s}} \exp \left[ -\frac{e(V_{4p} - V_{4s})}{kT_{ex}} \right].$$

(3.4)

The values $g_{4s}, g_{4p}$ are the degeneracy factors for 4s and 4p states of Ar-II, and the $V_{4s}, V_{4p}$ are the energy gaps between the ground state and 4s and 4p states of Ar-II respectively.

The distribution of the excitation temperature calculated with the number densities of ions in 4s and 4p levels is shown in Fig. 3.6. Specifically, $T_{ex}$ has a maximum of 3 eV around the cathode, and has relatively lower value of 1 – 2 eV near the anode. This qualitative tendency agrees with the measured electron temperature. It is notable that the calculated excitation temperature is close to the measured electron temperature, although quantitative argument may be inappropriate, because the accuracy of the excitation cross sections used here is at best a factor of 2 [73]. Comparing the calculated $T_e$ and $T_{ex}$, these are comparable around the cathode, which implies the validity of the assumption of LTE around the cathode. On the other hand, near the flared anode surface, $T_{ex}$ is less than $T_e$, thus the plasma is supposed to deviate from LTE because of the low plasma density. These results suggest that the measured electron temperature is underestimated near the anode surface, in other words, there is a possibility that the actual electron temperature near the anode surface is much higher.

### 3.4.4 Distribution of Excited Ions

The calculated distribution of number density of the excited ion in 4p level can be compared with the radiative intensity map shown in Ref. [14]. Strictly speaking, comparison between the numerical and the experimental result may not be appropriate, because the observed radiation (480.6 nm) is attributed to the transition between quartet states of 4p - 4s levels, and in our model, the interaction only among the doublet states are dealt with. Even then it will be useful to describe the distribution of the irradiating
excited ions in 4p level. As shown in Fig. 3.7, the calculated result shows that the 4p ions are distributed mainly around the cathode, and they rapidly decrease in the downstream region, which can be also seen from the measured intensity map. The calculated result has the maximum at the cathode tip probably owing to the compression by the Lorentz force, while the effect of the compression at the cathode tip cannot be seen obviously. This may be due to the difference in the cathode tip configuration. In addition, the decrease in the measured intensity on the cathode surface does not appear in the calculated result. The energy loss on the electrodes with regard to the electrons, which is ignored in the present model, may be related with this discrepancy.

From the populations of excited ions, radiation losses $Q_{rad}$ with regard to inter-level transfers of the doublet ions can be estimated by the following equation:

$$Q_{rad} = \int q_{rad} dV = \int \sum_{i>j} eV_{ij} A_{ij} n_i dV.$$ (3.5)

$Q_{rad}$ for $J = 8, 10, 12$ kA and the ratio of $Q_{rad}$ to the input power without energy deposition in a sheath are tabulated in Table 3.2. It can be seen that $Q_{rad}$ increases with input power, and amounts to 10% of input power for $J = 12$ kA. It has to be noted that these values should be interpreted as reference values, since only a few transfers are taken into account, and quartet ions are ignored in the present model.

On the whole, since the number densities of the excited ions ($\sim 10^{18}$ m$^{-3}$) are much less than those of the ions in ground state ($\sim 10^{21}$ m$^{-3}$), whether the CR model is incorporated in the model or not does not affect the overall flowfields and the performance significantly.
Fig. 3.5 Electron temperature (eV), 2D-MPDT, Ar, 1.25 g/s, $J = 12$ kA; (a) Calc., (b) Exp. [47].

Fig. 3.6 Calculated excitation temperature (eV), 2D-MPDT, Ar, 1.25 g/s, $J = 12$ kA.
3.4 Comparisons of Flowfields

Fig. 3.7 Distribution of excited ion in 4p level, 2D-MPDT, Ar, 1.25 g/s, \( J = 12 \) kA; (a) Calc. (m\(^{-3}\)), (b) Exp. [14] (The labeled values represent relative intensities of radiation).

Table 3.2 Radiation losses, 2D-MPDT, Ar, 1.25 g/s. V denotes a discharge voltage without a sheath voltage.

<table>
<thead>
<tr>
<th>( J ) (kA)</th>
<th>( JV ) (kW)</th>
<th>( Q_{\text{rad}} ) (kW)</th>
<th>( Q_{\text{rad}}/JV ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>64</td>
<td>2.2</td>
<td>3.4</td>
</tr>
<tr>
<td>10</td>
<td>101</td>
<td>6.7</td>
<td>6.6</td>
</tr>
<tr>
<td>12</td>
<td>151</td>
<td>16.0</td>
<td>10.6</td>
</tr>
</tbody>
</table>
3.5 Performance Evaluation

The purpose of this section is to compare the performance such as thrust, and thrust efficiency with the experimental results.

3.5.1 Thrust

The thrust $F$ is computed from a momentum flux at the thruster exit:

$$ F = \int_{\text{exit}} \left( \rho u^2 + p \right) dS. \quad (3.6) $$

Generally, the thrust evaluated via above equation includes both an electrothermal thrust and an electromagnetic thrust. A reaction force of the electrothermal thrust acts on the electrodes though the pressure, and a reaction force of the electromagnetic force acts on an electric circuit of the discharge. The component of the electromagnetic thrust $F_{em}$ can be computed by a volume integral of the Lorentz force in $x$ direction:

$$ F_{em} = \int \left( j \times B \right)_x dV. \quad (3.7) $$

Given an ideal condition such as a straight anode, a theoretical electromagnetic thrust can be derived as the following equation in which the effect of the Lorenz force in $x$ direction and the pressure force acting on the cathode surface are included [15]:

$$ F_{em} = \frac{H_0 d_a}{8W} J^2. \quad (3.8) $$

Here, $d_a$ denotes an interval between the upper and lower anodes. In Fig. 3.8, the total thrust $F$, the measured thrust [79], and the electromagnetic thrust $F_{em}$ are plotted, and the theoretical line of $F_{em}$ is also described. As the value of $d_a$ in Eq. (3.8), the anode interval at the inlet is substituted, because most of the discharge current concentrates around the upstream region. It can be seen that the calculation overestimates the total thrust about 1 N. The reason for this will be attributed to the fact that both side of the MPD thruster was normally covered by quartz glass during the thrust measurement, which will cause the thrust reduction via friction loss. In addition, the actual experimental apparatus will have the end effect at the both side of the thruster, which means deviation from the complete two-dimensional uniformity of the induced magnetic field in $z$ direction. Since the actual induced magnetic field at the side is less than that at the center of the thruster, the electromagnetic thrust will be reduced...
compared with the ideal condition. Regarding the electromagnetic thrust, the calculated value is in good agreement with the theoretical curve. Also, the result indicates that the electromagnetic thrust amounts to 65% of the total thrust at $J = 12$ kA, while it is 43% at $J = 8$ kA. Hence it can be said, in terms of the ratio of $F_{em}$ to $F$, the acceleration mode is changed from the electrothermal to the electromagnetic between $J = 8$ and 12 kA.

3.5.2 Thrust Efficiency

A thrust efficiency is defined as the ratio of a kinetic energy of the plasma at the thruster exit to an input power. With the thrust $F$, the thrust efficiency $\eta$ can be given with the following equation

$$\eta = \frac{F^2}{2\dot{m}J(V + V_{sh})}, \quad (3.9)$$

where $V$ and $V_{sh}$ denotes a voltage drop in the bulk plasma and a sheath voltage respectively. The voltage drop in the bulk plasma is given by a line integral of the electric field from the anode to the cathode. For $J = 8, 10, 12$ kA, $V$ is 8.0, 10.1, and 12.6 V respectively. Since the sheath effect is not included in the present model, we assume $V_{sh}$ to obtain the thrust efficiency. Although there are not comprehensive understandings on the sheath voltage drop in an MPD thruster, the sum of an anode sheath drop and a cathode sheath drop is supposed to range from 20 to 40 V [80].

In Fig. 3.9, the thrust efficiencies of the numerical and the experimental results are plotted. Regarding the numerical results, the closed square plots are given under the condition of $V_{sh} = 30$ V, and the top and the bottom of the error bar corresponds to the thrust efficiency for $V_{sh} = 20$ and 40 V respectively. It can be seen that, although qualitative feature of the thrust efficiency is well captured, the numerical results overestimate the experimental values due to the difference in the thrust $F$ as mentioned above. Also, the calculated voltage drop in the bulk plasma may be underestimated. However, there is a possibility for the calculation to underestimate the voltage drop in the bulk plasma, since the effect of anomalous resistivity is ignored in the present model [58].
Fig. 3.8 Thrust vs. discharge current, 2D-MPDT, Ar, 1.25 g/s, Experimental values are referred from [79].

Fig. 3.9 Thrust efficiency vs. discharge current, 2D-MPDT, Ar, 1.25 g/s, The top and the bottom of the error bar for the calculation results represents the thrust efficiency for the sheath voltage $V_{sh} = 20$ and $40$ V respectively. Experimental values are referred from [79].
3.6 Summary

A numerical code with the detailed modeling to simulate the flowfields of a self-field two-dimensional magnetoplasmadynamic thruster has been developed, and comparisons of the simulated plasma flow and the performance with the experimental results are conducted. In the physical model, non-equilibrium ionization of argon propellant and the collisional-radiative model for Ar-II are incorporated in order to examine the detailed reaction processes.

The results show that the current path is obliquely skewed in the thruster due to the Hall effect, which can be seen in the measured data, although the distortion of the calculated current path in the vicinity of the flared anode surface appears excessively. The fan-shaped distribution of the electron number density is well captured by the calculation, although quantitative argument remains. The distribution of the electron temperature, however, differs qualitatively from the measured result obtained from a relative intensity method, especially near the flared anode surface. Then the excitation temperature is numerically estimated from the calculated populations of the excited ions in 4s and 4p states, which is to be compared with the measured electron temperature. Consequently, it is shown that the calculated excitation temperature becomes close to the measured result, which suggests that the plasma deviates from local thermodynamic equilibrium (LTE) near the flared anode surface, while the LTE assumption seems valid around the cathode.

The qualitative characteristics of the computed thrust and thrust efficiency against variation of the discharge current are almost the same with the experimental result, although the calculated values are slightly overestimated.

It is has to be noted that the measured distributions are not in steady-state but time-averaged results during the pulsed discharge about 500 µs, which can lead to some difference between the calculated and experimental results.
Chapter 4.

Plasma Behavior and Performance around Critical Current

In this chapter, plasma behavior around a critical current, around which appearance of acceleration mode transition from electrothermal to electromagnetic mode is believed, is focused on. Under the same mass flow rate, the discharge current is varied around the estimated critical current to examine the effect of the current magnitude on the plasma flowfield and the performances. Above a critical current, it is inferred that current-carrier shortage, so-called anode starvation, occurs near the anode surface. This study aims to quantitatively investigate the phenomenon of anode starvation. In addition, the effect of segmented anodes on the suppression of anode starvation is also argued.

4.1 Thruster Configuration

The thruster used in this section consists of a short cathode and a flared anode, as shown in Fig. 4.1. The cathode is 8 mm in diameter and 13 mm in length. The inner diameter of the anode is 28 mm at the inlet and 56 mm at the outlet. The vertical wall located at $z = 45$ mm is composed of an insulating material. This configuration is based on the MPDT whose performance characteristics, such as thrust and voltage, have been examined by Nakata [81]. Therefore, we can directly compare our results with the experimental results.

We adopt a structured grid with dimensions of $110 \times 35$ in the thruster region ($z \leq$
4.2 Calculation Conditions

The propellant injected from the inlet is argon, and the prescribed mass flow rate is 0.8 g/s. Since the consideration of the plasma ignition at the inlet is difficult to incorporate into the numerical simulation, the processes are ignored and a relatively high temperature and high ionization fraction are assumed for the plasma inflowing into the thruster. The respective temperatures of the heavy particles and the electrons at the inlet are assumed to be 8,000 K and 10,000 K, and we set the ionization fraction at the inlet to 0.5.

Under the present mass flow rate and the configuration, the critical current estimated from Eq. (1.10) is about 6 kA. Thus, the discharge current is varied from 4 kA to 8 kA in order to examine the behavior of the transition around the critical current. For the present mass flow rate, the parameter $J^2/m$, which is widely known as a key parameter in MPDTs, can be varied from 20 to 80 (kA)$^2$/gs$^{-1}$. According to the results from past experiments with argon propellant conducted by Yoshikawa [23], the critical value of

![Fig. 4.1 Thruster configuration and computational grid (Axisymmetric flow).]
the parameter $J^2/\dot{m}$, which we denote as $J^2/\dot{m}_c$, at which the onset phenomenon occurs is 56 (kA)$^2$/gs$^{-1}$ for $\dot{m} = 0.6$ g/s and 36 (kA)$^2$/gs$^{-1}$ for $\dot{m} = 3$ g/s. Thus

According to the experimental results regarding the radial electron temperature distribution at the outlet under similar operational conditions obtained by Tahara [82], the electron temperature near the wall is estimated to be approximately between 1 and 2 eV. Thus, the temperature $T_e$ of the electrons on the walls is limited to less than 2 eV. For comparison, the cases involving adiabatic conditions for the electron temperature are also calculated in order to consider the influence of $T_e$ on the wall. We denote the former condition as “limited isothermal $T_e$ condition”, and the latter as “adiabatic $T_e$ condition”.

In order to examine the sensitivity of the results to the changes in the boundary conditions, we calculate the changes in thrust under various conditions at $J = 6$ kA. When the heavy particle temperature at the inlet is set to a lower value of 5,000 K, the thrust is reduced by about 1%. When the condition that $T_e < 1$ eV is imposed on the walls, the thrust is reduced by about 2%. Therefore, it is considered that the thrust is insensitive to the particular values set at the boundaries, and this is also true for the other results.

<table>
<thead>
<tr>
<th>Table 4.1 Calculation Conditions.</th>
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</thead>
<tbody>
<tr>
<td>Propellant</td>
</tr>
<tr>
<td>Mass flow rate (g/s)</td>
</tr>
<tr>
<td>Discharge current (kA)</td>
</tr>
<tr>
<td>Inlet heavy particle temperature (K)</td>
</tr>
<tr>
<td>Inlet electron temperature (K)</td>
</tr>
<tr>
<td>Ionization fraction at inlet</td>
</tr>
<tr>
<td>Electron temperature on walls</td>
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</table>

4.3 Flowfields of Coaxial Self-Field MPD Thruster

In this section, the flow fields around the critical current are discussed. All figures shown here present results obtained under the limited isothermal $T_e$ condition.
4.3 Flowfields of Coaxial Self-Field MPD Thruster

4.3.1 Current Path

In order to examine the change in the flow field around the critical current, the flow fields for \( J = 4 \) and 7 kA are compared. Figure 4.2 shows the current path, which is equivalent with the contour lines of the function \(|rB|\). The labeled values on the contour lines denote the ratio of the current flowing downstream viewed from the line to the total discharge current. For \( J = 4 \) kA, the numerical results indicate that the current does not spread out of the thruster, and no current spot is found on the electrodes, i.e., the current distribution on the electrodes is diffused. On the other hand, when \( J = 7 \) kA, which is above the critical current, the current path is drastically different from that in the case of \( J = 4 \) kA, where the current lines concentrate at the anode edge (\( z = 45 \) mm, \( r = 28 \) mm) and the cathode root (\( z = 0 - 6 \) mm, \( r = 4 \) mm). Although the current lines whose labels are less than 0.6 tend to concentrate near the anode edge for \( J = 7 \) kA, the current lines above 0.6 are similar in the two cases (\( J = 4 \) kA and 7 kA). Around the middle part of the flared anode surface (\( z \approx 25 \) mm), the electrons are prevented from flowing into the anode, and are carried downstream. The obliquely skewed current lines are attributed to the Hall effect.

4.3.2 Hall Parameter

In Fig. 4.3, the respective Hall parameter distributions at \( J = 4 \) and 7 kA are described. When \( J = 4 \) kA, the Hall parameter is less than unity in the discharge chamber, and it is highest at the cathode root since the induced magnetic flux density is largest there. Since the Hall parameter is relatively low inside the thruster, a noticeable distortion of the current lines due to the Hall effect does not appear for \( J = 4 \) kA. On the other hand, in the case of \( J = 7 \) kA, the current distribution changes into a different one, as shown in Fig. 4.2-(b). At the cathode root, the Hall parameter is about 6. However, it decreases steeply down to 1 - 2 around the downstream region near the cathode surface due to the appearance of a high-density region, as shown below. Moreover, it should be noted that the region where the Hall parameter is high appears around the anode surface of the nozzle (diverging part). The highest value of the Hall parameter amounts to 11.4 around \( z = 28 \) mm, \( r = 20 \) mm, while the Hall parameter at the same location in the case of \( J = 4 \) kA is about 0.1. Since the electrons in the region where the Hall parameter is
high are forced to flow downstream by the Hall effect, they are prevented from flowing into the anode. As a result, the current lines around the region tend to be parallel to the anode surface, which leads to the intensive concentration of current at the anode edge. A steep increase in the Hall parameter in the vicinity of the anode surface for an increasing current around the critical current was also observed in an MPDT experiment conducted by Diamant et al. at Princeton [83].

![Fig. 4.2 Current path, Ar, 0.8 g/s; (a) 4kA, (b) 7 kA. The labeled values on the contour lines denote the ratio of the current flowing downstream viewed from the line to the total discharge current.](image)
4.3 Flowfields of Coaxial Self-Field MPD Thruster

Fig. 4.3 Hall parameter, Ar, 0.8 g/s; (a) 4 kA, (b) 7 kA.
4.3.3 Number Density of Heavy Particles

The high Hall parameter around \( z = 28 \text{ mm}, r = 20 \text{ mm} \) is attributed to the low number density in that region. Figure 4.4 describes the number density of heavy particles in both cases \((J = 4 \text{ and } 7 \text{ kA})\). At \( J = 4 \text{ kA} \), the number density at \( z = 28 \text{ mm}, r = 20 \text{ mm} \) is about \( 1.7 \times 10^{21} \text{ m}^{-3} \). When \( J = 7 \text{ kA} \), however, the number density of heavy particles is about \( 1.6 \times 10^{20} \text{ m}^{-3} \). Since the Hall effect leads to the current profile being parallel to the anode surface, the Lorentz force perpendicular to the anode surface provokes the density depletion.

We can also see from Fig. 4.4-(a) that a compression wave extends from the cathode tip. At \( J = 4 \text{ kA} \), the weak compression wave propagates toward the anode surface and is reflected around \( z = 30 \text{ mm} \). After that, the reflected compression wave propagates toward the axis of symmetry, which leads to the density distribution in the plume region shown in Fig. 4.4-(a). On the other hand, the structure of the density distribution in the case of \( J = 7 \text{ kA} \) differs from that in the case of \( J = 4 \text{ kA} \). There is a compression wave for \( J = 7 \text{ kA} \) starting from the cathode root. It is notable that the direction of the compression wave is slightly varied around \( z = 20 \text{ mm}, r = 9 \text{ mm} \) due to another compression wave extending from the vicinity of the cathode tip. Although the combined compression wave reaches the anode surface, there is no reflected compression wave in the plume region.

4.3.4 Plasma Beta

The ratio of the static pressure to the magnetic pressure, namely beta value of the plasma, serves to understand the plasma depletion near the anode:

\[
\beta_p = \frac{p}{\frac{B^2}{2\mu_0}}. \tag{4.1}
\]

The colored regions in Fig. 4.5 correspond to the area where \( \beta_p \) is less than unity. The solid and dashed lines are the boundary of \( \beta_p = 1 \) for \( J = 4 \text{ kA} \) and \( 7\text{ kA} \) respectively. Whereas the \( \beta_p \) is more than unity in almost whole region for \( J = 4 \text{ kA} \), the large area where \( \beta_p < 1 \) appears for \( J = 7 \text{ kA} \) due to the current expansion toward the downstream region. This result suggests that the magnetic pressure, which pushes the plasma toward the symmetric axis, is dominant near the anode.
4.3 Flowfields of Coaxial Self-Field MPD Thruster

Fig. 4.4 Number density of heavy particles (m⁻³), Ar, 0.8 g/s; (a) 4 kA, (b) 7 kA.
4.3.5 Velocity

Figure 4.6 shows the velocity distribution and the streamlines for $J = 7$ kA. The plasma is accelerated toward the axis of symmetry, after which it is compressed, leading to a high-velocity cathode jet as a result of the so-called pumping force. The current profile near the cathode root shown in Fig. 4.2-(b) causes the strong compression of the plasma onto the cathode surface since the magnitude of the Lorentz force is the highest near the cathode root.
4.4 Carrier Shortage on Anode Surface

In the previous section, we have shown that the plasma density near the anode surface is drastically decreased for currents higher than $J_c$. This plasma depletion on the anode surface is considered to result in a shortage of current carriers. The theory addressing the issue of carrier shortage, the so-called anode starvation theory, utilizes the ratio of the net current density perpendicular to the anode surface (which we denote as $j$) to the electron thermal current $j_{th} = \frac{e n_e C_e}{4}$ as an indicator of the carrier shortage, where $C_e$ denotes the electron thermal speed $\sqrt{\frac{8kT_e}{\pi m_e}}$. If $j$ is less than $j_{th}$, an ion-rich sheath forms so that the flow of electrons to the anode is prevented (Fig. 4.7). On the other hand, if $j$ exceeds $j_{th}$, an electron-rich sheath forms which drives the electrons to the anode due to the lack of electrons as current carriers, which leads to a large energy input on the anode surface and significant evaporation of the anode material. Regarding the experimental evidence of the above prediction, Hügel and Kurtz specified that the voltage drop at the anode changes its sign around the critical current $[18,34]$. In addition, Gallimore and Diamant showed that anode starvation is closely related to the Hall parameter $[83,84]$. In order to discuss this phenomenon, we describe the current density perpendicular to the anode and the ratio of the current density to the electron thermal current at the anode edge ($z = 45$ mm, $r = 28$ mm) as a function of discharge current in Fig. 4.8. Since the respective current densities for $J = 4$ and 5 kA are low, the ratio $j/j_{th}$ is also kept low. At $J = 6$ kA, which approximately corresponds to the critical current, $j/j_{th}$ begins to increase, and at $J = 7$ kA, the ratio $j/j_{th}$ under the limited isothermal $T_e$ condition rises steeply to about unity due to both the concentration of the current and the depletion of the gas density on the anode surface. When the discharge current is increased further, the $j/j_{th}$ value keeps rising, taking a value of about 1.9 at $J = 8$ kA. It is noted that this tendency can be seen even under the adiabatic $T_e$ condition, although the rate of increase is lower. This abrupt rise of the $j/j_{th}$ value does not appear unless the Hall effect is taken into account.

There are analytical expressions for $J_c$ in terms of the anode starvation theory $[85-87]$. Baksht investigated the anode starvation analytically under the assumption that the radial current is equal to electron thermal current, and derived the following relational expression for the critical current $[85]$.
where the unit is cgs-Gauss. The values of $c$, $R$, $d$, and $L$ are the light velocity, mean channel radius, the electrode gap, and the channel length, respectively, where $R \gg d$ is assumed. We discuss the critical current given by Baksht’s expression in Eq. (4.2), although $R \gg d$ is not satisfied in our case. If we set the parameters to $d = 17$ mm (average value), $L = 45$ mm, $R = 9$ mm, $T = 2$ eV, and $\sigma = 5000$ S/m, Baksht’s critical current is estimated as 5.9 kA. Although this value depends on the input parameter, $J_c$ varies between 5 and 7 kA. It is notable that the critical current presumed from our results is almost the same as the value for $J_c$ obtained by Baksht’s model.

Fig. 4.7 Relationship among current, electron thermal current, and potential near anode surface.
A comparison of the results for the thrust obtained from our numerical experiment with those from actual experiments is essential for confirming the validity of our code. Figure 4.9 shows our numerical results and those from the experiments performed by Nakata [81]. The theoretical electromagnetic thrust, which is obtained from Maecher’s formula (Eq. (1.8)) is also depicted in the figure. If electromagnetic thrust is dominant, the total thrust is considered to approach the theoretical electromagnetic thrust. An approximate numerical evaluation of the thrust is obtained from the momentum flux at the thruster exit:

\[ F = \int_{\text{exit}} \left( \rho u^2 + p \right) dS. \]  

(4.3)

We can see that the numerical results for the thrust under the limited isothermal \( T_e \) condition are in good agreement with the experimental ones over the entire current range, which indicates that our results are plausible. Figure 4.9 indicates that the numerical and the experimental results tend to approach the theoretical curve of \( d = 3/4 \). Although the thrust in the case of the adiabatic \( T_e \) condition might be overestimated, the
deviation from the experimental results is about 1 N at most.

Under the condition of the present mass flow rate of 0.8 g/s and the above mentioned thruster configuration, the critical current $J_c$ given by Eq. (1.10) is about 6 kA, when the parameter $d$ is set to 3/4. This fact predicts that the acceleration is electrothermal for $J < 6$ kA, and that electromagnetic thrust is dominant for $J > 6$ kA. According to Fig. 4.9, the numerical results under the limited isothermal $T_e$ condition almost coincide with the theoretical curve above 7 kA, whereas the numerical results differ from the theoretical ones for $J < 6$ kA.

![Fig. 4.9 Thrust as a function of current, Ar, 0.8 g/s (The “limited isothermal” and “adiabatic” denote the limited isothermal $T_e$ condition and adiabatic $T_e$ condition respectively. Experimental data are from Ref.[81]).](image)

### 4.6 Voltage-Current Characteristics

It has been recognized as a common feature that the transition from electrothermal to electromagnetic acceleration entails some changes in the voltage-current characteristics. If a discharge current is lower than the critical current $J_c$, i.e., in the electrothermal acceleration mode, the voltage is linearly proportional to $J$. At discharge currents higher than $J_c$, however, the voltage deviates from the linear relation. At the transitional current,
the voltage corresponding to the work performed by the Lorentz force is considered to dominate the Joule heating component. When the electromagnetic force is dominant, the voltage should be proportional to the cube of the current as

\[ JV \approx F_{en} u_c = b J^2 \frac{bj^2}{m} \Rightarrow V \approx \frac{b^2 J^3}{m}. \] (4.4)

However, the simulation result, which ignores the Hall effect, suggests that the work performed by the Lorentz force is not the main cause for the steep voltage increase [88]. Figure 4.10 describes the voltage-current characteristics obtained from our numerical results, as well as the experimental results. The results where the Hall effect is not considered are also shown in the figure. The voltage is given by a line integration of the electric field between the anode and the cathode:

\[ V = \int_a^c E \cdot dl = \int_a^c \left( \frac{j}{\sigma} - u \times B + \frac{1}{en_c} j \times B \right) \cdot dl. \] (4.5)

Note that this equation does not take into account the sheath voltage, and thus the calculated voltages shown in Fig. 4.10 must be lower than the experimental results by about 20 – 40 V [80]. At currents above \( J_c \), the numerical results calculated by taking into account the Hall effect are significantly higher than those without the Hall effect, which indicates that the Hall effect plays an important role in the steep voltage increase above \( J_c \). This property seems to be consistent with the experimental results. The slope of the V-J curve under the adiabatic \( T_e \) condition is similar to that obtained in the experiment. Regarding the numerical results, the fitted curve of the plots for 7, 8 kA indicates that \( V \propto J^{2.9} \) under the limited isothermal \( T_e \) condition and \( V \propto J^{2.8} \) under the adiabatic \( T_e \) condition. Interestingly, these results show that our numerical results with the Hall effect support the theoretical presumption \((V \propto J^3)\). It should be noted that these results are obtained when the Hall effect was incorporated, although an anode sheath potential drop, which might grow in the range of \( J > J_c \), is not taken into account. This steep rise of the voltage is discussed in detail in the next section.

Based on the calculated thrust and voltage, the thrust efficiency can be estimated. We evaluate the thrust efficiency following equation:
\[
\eta = \frac{F^2}{2\bar{m}J(V + V_{sh})}.
\]

(4.6)

The thrust performance at \( J = 6 \) kA under the limited isothermal condition is described in Table 4.2. For the evaluation of the thruster efficiency, we assumed a constant sheath voltage \( V_{sh} = 20 \), and 30 V. The thrust efficiency obtained from the experiment is about 13% at \( J = 6.5 \) kA, and 12% at 5.1 kA. The error of several percent for the thrust efficiency between the calculated and experimental results will be due to uncertainty on the sheath voltage, and/or the effect of anomalous resistivity [58]. The thrust efficiency at \( J = 8 \) kA under the limited isothermal condition for \( T_e \) is decreased to 15.5% \((V_{sh} = 20 \) V\) due to the excessive increase in the discharge voltage, while the thrust efficiency is monotonically increased in the experiment even above 6 kA [81].

Fig. 4.10 Voltage – Current characteristics, Ar, 0.8 g/s (The “limited isothermal” and “adiabatic” denote the limited isothermal \( T_e \) condition and adiabatic \( T_e \) condition respectively. Calculated data do not include the sheath voltage. Experimental data are from Ref. [81].).
4.7 Increase in Discharge Voltage

In order to clarify the relation between the discharge voltage and acceleration modes, $V_J$ as the voltage with regard to Joule heating, and $V_{em}$ as a voltage with regard to the work of the electromagnetic force are defined as follows:

$$JV_J = \int \frac{J^2}{\sigma} dV,$$

(4.7)

$$JV_{em} = \int \mathbf{u} \cdot (\mathbf{j} \times \mathbf{B}) dV = JV - JV_J.$$

(4.8)

Here, $dV$ denotes an infinitesimal volume. $V_J$ and $V_{em}$ are presented in Table 4.3 for the results of the adiabatic $T_e$ condition, where the values in parentheses represent the results without the Hall effect. The ratio of an electromagnetic thrust to the total thrust $F_{em}/F$ is also shown. For $J = 4$ kA, there is no remarkable difference between the results with and without the Hall effect due to the low value of the Hall parameter, as shown in Fig. 4.3. It should be noted that the contribution of $V_{em}$ to $V$ ($V_{em}/V$) is 20% at most, which leads to a low percentage for the electromagnetic thrust (33%), i.e., the thrust is in electrothermal mode. On the other hand, for $J = 7$ kA, each voltage value for which the Hall effect is considered is twice as large as that without the Hall effect. Since the Hall effect causes the extension of the current path, $V_J$ (the Joule heating component) is increased. The current extension due to the Hall effect also serves to increase $V_{em}$. Comparing the result for $J = 7$ kA with that for $J = 4$ kA, the rate of increase for $V_{em}$ is higher than that for $V_J$, which results in a relatively high $V_{em}/V$ ratio of 37-40%. This in turn contributes to the high $F_{em}/F$ ratio of 65-69%, i.e., the thrust is in electromagnetic

<table>
<thead>
<tr>
<th>Discharge Current (kA)</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass Flow Rate (g/s)</td>
<td>0.8</td>
</tr>
<tr>
<td>Thrust (N)</td>
<td>8.47</td>
</tr>
<tr>
<td>$I_{eq}$ (s)</td>
<td>1080</td>
</tr>
<tr>
<td>Discharge Voltage (V)</td>
<td>27.1</td>
</tr>
<tr>
<td>(except for sheath voltage)</td>
<td></td>
</tr>
<tr>
<td>Thrust Efficiency (%)</td>
<td>15.7 ($V_{sh}=20V$)</td>
</tr>
<tr>
<td></td>
<td>12.9 ($V_{sh}=30V$)</td>
</tr>
</tbody>
</table>

Table 4.2 Typical thruster performance (Discharge voltage does not include sheath voltage. Thrust efficiency is evaluated under the assumption of $V_{sh} = 20, 30$ V).
mode. However, it should be noted that $V_J$ is larger than $V_{em}$ even for $J = 7$ kA, i.e., both $V_J$ and $V_{em}$ contribute to the characteristics of $V \propto J^{2.8}$ (for adiabatic $T_e$ condition). The high $F_{em}/F$ rate, in spite of the fact that $V_J > V_{em}$, indicates that a large amount of thermal energy is lost as both ionization energy and thermal conduction to the walls.

Table 4.3 Summary of voltages and ratio of electromagnetic force to total thrust. The values in parentheses represent results without the Hall effect. (Adiabatic $T_e$ condition is adopted.)

<table>
<thead>
<tr>
<th>Current</th>
<th>$V$ (V)</th>
<th>$V_J$ (V)</th>
<th>$V_{em}$ (V)</th>
<th>$V_{em}/V$ (%)</th>
<th>$F_{em}/F$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 kA</td>
<td>11.6</td>
<td>9.8</td>
<td>1.8</td>
<td>15.7</td>
<td>33.4</td>
</tr>
<tr>
<td></td>
<td>(9.6)</td>
<td>(7.7)</td>
<td>(1.9)</td>
<td>(19.5)</td>
<td>(33.6)</td>
</tr>
<tr>
<td>7 kA</td>
<td>31.7</td>
<td>20.0</td>
<td>11.7</td>
<td>36.9</td>
<td>65.2</td>
</tr>
<tr>
<td></td>
<td>(15.4)</td>
<td>(9.2)</td>
<td>(6.2)</td>
<td>(40.3)</td>
<td>(69.1)</td>
</tr>
</tbody>
</table>

4.8 Segmented Anode

Although the anode density depletion can be circumvented by reducing the nozzle angle or by changing the electrode size, such measures also affect the thrust characteristics (the thrust is expected to decrease). In order to avoid the carrier shortage without changing the geometry of the thruster, we employ a segmented anode as shown in Fig. 4.11. Let us denote each anode as A1 and A2, respectively, and assume that an external electrical circuit can divide the discharge current equally between the two anodes, i.e. $J_{A1} : J_{A2} = 1:1$. We can adjust the magnitude of the current at each electrode by using two half-scale power sources for each anode, or by employing ballast resistances with one power source. The anode is segmented by placing an insulator at $z = 20-30$ mm. With this geometry, we can expect not only the elimination of the current concentration around the thruster exit, but also the suppression of the plasma density depletion on the nozzle surface without reducing the nozzle angle. The segmented anode technique was employed by Kurtz for the straight anode configuration, although the current at each electrode was not prescribed [18]. The magnetic flux density along the insulated part between A1 and A2 is given by Ampère's law, i.e. $B_\theta = -\mu_0 J_{A2} / 2\pi r$. The limited isothermal condition is adopted as the boundary condition for the electron temperature.

Although the values of the thrust and the thrust efficiency for the continuous anode
and the segmented anode were almost the same \((F = 8.4 \text{ N, } \eta = 15.3\% \ (V_{th} = 20))\), the flow and field structures are different in each case. In Fig. 4.12, the current path for \(J = 7 \text{ kA}\) is described. Since the discharge current of A1 does not expand significantly toward the downstream region, the current density at the edge of A2 is moderated to \(4.2 \times 10^6 \text{ A/m}^2\), whereas it is \(6.0 \times 10^6 \text{ A/m}^2\) in the case of the continuous anode. In comparison to the continuous anode case, the current along the insulator surface, and hence the radial Lorentz force, is reduced in the case of the segmented anode. Thus, plasma depletion around the anode nozzle surface is not observed. Figure 4.13 shows the number density of heavy particles along the anode nozzle surface from \(z = 10\) to 45 mm. In the case of the continuous anode, the density decreases monotonically due to the nozzle expansion in the region of \(z < 28\) mm. Since the flow near the anode surface is subsonic, the density is recovered at \(z \approx 28\) mm due to the backward propagation of the disturbance induced by the compression wave extending from the cathode tip. A decrease in the density is not observed at the end of A1 for the segmented anode.

Figure 4.14 shows the respective values of \(j/j_{th}\) at the edge of A1 and A2 for \(J = 7\) and 8 kA. At \(J = 7 \text{ kA}\), the \(j/j_{th}\) value at the edge of A2 decreases from 1 to 0.5 in the case of the segmented anode. This is due to a low current density, and hence the weak Lorentz force, which leads to a higher gas density in comparison to the continuous anode. Even at the edge of A1, the \(j/j_{th}\) value is below 0.6. Therefore, it is considered that a segmented anode can be useful for preventing high current concentration at the anode edge.
Chapter 4 Plasma Behavior and Performance around Critical Current

Fig. 4.11 Segmented anodes.

Fig. 4.12 Current path with segmented anodes, Ar, 0.8 g/s, $J = 7$ kA.
Fig. 4.13 Comparison of number density of heavy particles along the nozzle part with segmented and continuous anode, Ar, 0.8 g/s, $J = 7kA$.

Fig. 4.14 Ratio of current density to electron thermal current at the anode edge for segmented and continuous anode, Ar, 0.8 g/s (limited isothermal $T_e$ condition).
4.9 **Summary**

A numerical examination of the plasma behavior in a coaxial self-field MPD thruster (several hundred of kilowatts class) is examined over a wide operation range around the critical current for an argon mass flow rate of 0.8 g/s. In order to compare the respective plasma flows in electrothermal and electromagnetic acceleration modes, two situations are compared; $J = 4$ and 7 kA, where the critical current is about 6 kA.

Along the anode surface, the Hall parameter increases together with the current, leading to an obliquely skewed current profile at discharge currents higher than the critical current. In this case, an increase in the radial component of the Lorentz force induces the depletion of the gas density around the anode surface, which corresponds to a shortage of current carriers, so-called anode starvation. A steep increase in the ratio of the current density to the electron thermal current at the anode edge around the critical current indicates the occurrence of a carrier shortage. The thrust is relatively in good agreement with the measured value. The voltage begins to rapidly increase above the critical current due to increase in both Joule heating and work by Lorentz force. It is also shown that the Hall effect is predominantly attributed to the steep increase in the voltage around the critical current.

In order to suppress the density depletion at the anode surface, a segmented anode is employed in the numerical simulation. The segmented anode leads to the suppression of the current concentration and the enhancement of the number density at the anode. A segmented anode is therefore considered suitable for realizing a stable discharge for currents above the critical current.
Chapter 5.

Attempt toward Low-Power Operation Modes

In this chapter, two subjects are discussed; a pulsed SFMPDT, and an applied-field MPD thruster (AFMPDT). The pulsed SFMPDT aims to reduce time-averaged input power with a repetitive pulsed mode of 1-10 Hz. The time-averaged input power can be widely varied by changing a duty ratio of a pulsed discharge. In this study, the effect of current waveforms and propellant species on transient plasma behavior and performances are focused on. The AFMPDT attempts to obtain additional acceleration of plasma with a lower power level than SFMPDTs by applying a magnetic field. Fundamental flow properties under the existence of an applied magnetic field, thrust production mechanisms, and energy conversion processes are discussed in detail.

5.1 Pulsed Self-Field MPD Thruster

In this study, transient plasma behavior and performance of a pulsed SFMPDT is numerically examined. Transient flow properties of pressure, ionization fraction, and current path are shown for various current waveforms. One of key issues for pulsed operation will be how to enhance performances, in particular a specific impulse, hence the effect of propellant species on the flowfields and the performance are also discussed.
In this thesis, argon and helium are adopted.

5.1.1 Current Waveforms

A typical current waveform employed in experiments is described in Fig. 5.1. It can be seen that the characteristic time for current rise is several tens of microseconds. Actually, a current waveform is generally specified by a variety of parameters such as a circuit constant and plasma resistivity. However, in this study, the current waveforms are assumed by the following simple function for simplicity:

$$ J(t) = J_{\text{max}} \left( 1 - \exp \left( -\frac{t}{\tau} \right) \right). $$

(5.1)

$J_{\text{max}}$ is the maximum current, and $\tau$ denotes a discharge time constant which corresponds to characteristic time for current rise (Fig. 5.2). The duration time of a discharge is denoted as $T_{\text{on}}$. By varying these parameters, an arbitrary current waveform can be simulated. However, just after the ignition, the discharge current derived from Eq. (5.1) is quite low, which makes it difficult to numerically solve the induction equation. Therefore, in this study, the discharge current is set to 100 A only if $J(t)$ of Eq. (5.1) is less than 100 A. For simplicity, only a current-rising phase of a single shot is argued, i.e. a current-descending phase is not simulated.

![Fig. 5.1 Actual discharge current waveform [89].](image-url)
5.1 Pulsed Self-Field MPD Thruster

5.1.2 Parameters for Pulsed Operation

The description of the performance parameters to evaluate an electric propulsion device in Subsec. 1.1.2 is specialized for steady-state operation, thus we have to rewrite them to include the concept of a pulsed operation. Thrust and mass flow rate are to be replaced with an impulse bit $I_b$ and a massshot $\Delta m$ as

\[ I_b = \int_0^{T_{on}} F \, dt, \quad \Delta m = m_0 + mT_{on}, \] (5.2)

where $m_0$ is the total mass of propellant existing in a thruster at ignition. Since a current-descending phase ($T > T_{on}$) is not considered, the definition of $I_b$ neglects the residual impulse after the end of discharge. The input energy during a discharge is given by

\[ \Delta E = \int_0^{T_{on}} (H_{in} + J(V + V_{sh})) \, dt. \] (5.3)

The $H_{in}$ and $V_{sh}$ denote an intrinsic enthalpy flux at the inlet and a sheath voltage. The discharge voltage $V$ is computed by a line integration of the electric field. Since the sheath voltage is difficult to evaluate, the sheath voltage is assumed to be constant during a discharge in this study for simplicity. With the parameters defined above, specific impulse and thrust efficiency can be defined as follows:

---

Fig. 5.2 Assumed current waveform with Eq. 5.1.
\[ I_{sp} = \frac{I_b}{\Delta m g}, \quad \eta = \frac{I_b^2}{2\Delta m g} = \frac{g}{2} I_{sp} \frac{I_b}{\Delta E}. \]  

(5.4)

The ratio \( I_b/\Delta E \) is called as momentum coupling coefficient which corresponds to thrust-power ratio in Eq. (1.4). As Eqs. (5.2) and (5.3) indicate, all parameters for performance evaluation are altered by changing the duration time \( T_{on} \) even for a constant mass flow rate.

### 5.1.3 Thruster Configuration

Thruster configuration and corresponding computational region employed here are described in Fig. 5.3. The thruster has a coaxial structure with a flared anode and a short cathode. Computation of the plume region is omitted for simplicity. Since the discharge current may concentrate at the anode edge as inferred from the previous chapter, an insulator is equipped at the end of the anode to confine current-concentrated region within the computational domain. It is assumed that propellant is injected from a port located at the inlet (\( z = 0 \) mm, \( r = 7-11 \) mm).

![Fig. 5.3 Thruster configuration and computational grid. The propellant is injected from a port located at the inlet (\( z = 0 \) mm, \( r = 7-11 \) mm)](image)

### 5.1.4 Initial Conditions

When one attempts to simulate the temporal evolution of a pulsed discharge, an appropriate initial condition has to be assumed at the initiation of discharge. In reality,
cold gas is injected into a discharge chamber at first, and after that a discharge is to initiate. Since considerable delay of initiation of discharge after starting propellant injection will waste large amount of propellant, the synchronous of propellant injection and discharge ignition will be sensitive for specific impulse. However, some assumption will be necessary for the initial condition to model the complex phenomena. In this study, the initial flow condition is assumed to be a steady flow injected at 300 K into the thruster.

At ignition, initial distribution of current path has to be set, i.e. an initial magnetic field distribution has to be computed by solving the induction equation in a steady form with an iterative method. Then the electron temperature and the ionization fraction are assumed to be empirically selected low values, and to be uniform in the thruster. From a phenomenological point of view, these assumptions of uniform electron temperature and ionization fraction at $t = 0$ s may be unrealistic. However, these assumptions are insensitive to the performance, because the electron temperature is rapidly decreased in the region that little discharge current flows. The initial electron energy transferred to heavy particles via collision hardly increases heavy particle temperature.

5.1.5 Effects of Discharge Time Constant

**Calculation Conditions**

The effects of discharge time constant $\tau$ are evaluated under the conditions of a constant Ar mass flow rate ($\dot{m} = 1.2$ g/s), and constant duration time ($T_{on} = 250$ µs). In addition, the maximum current $J_{\text{max}}$ is set to the critical current for the given mass flow rate. In the estimation of critical current, $d = 3/4$ is used in Eq. (1.10). The parameters needed for simulation are listed in Table 5.1. The current rising time is varied by changing the discharge time constant $\tau$ from 1 to 20 µs. The time variation of the current waveform is shown in Fig. 5.4.

Inlet conditions are summarized in Table 5.2. Among these inlet parameters, the inlet temperature is difficult to determine during a time-developing discharge, and the ionization fraction as well. Thus, in this study, the inlet temperature and the ionization fraction are assumed to be constant during the discharge. At the inlet, the heavy particle
temperature, the electron temperature, and the ionization fraction are assumed to be 5,000 K, 8,000 K, and 0.1 respectively. It has been confirmed that the simulation results such as the thrust and voltage are insensitive to the inlet conditions. The sheath voltage necessary for the evaluation of input energy is assumed to be 20 V.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\dot{m}$ (g/s)</th>
<th>$J_{\text{max}}$ (kA)</th>
<th>$\tau$ (µs)</th>
<th>$T_{\text{in}}$ (µs)</th>
<th>$\Delta E$ (J)</th>
<th>$\Delta m$ (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2</td>
<td>7.2</td>
<td>1</td>
<td>250</td>
<td>75.5</td>
<td>0.453</td>
</tr>
<tr>
<td>2</td>
<td>1.2</td>
<td>7.2</td>
<td>10</td>
<td>250</td>
<td>72.4</td>
<td>0.453</td>
</tr>
<tr>
<td>3</td>
<td>1.2</td>
<td>7.2</td>
<td>20</td>
<td>250</td>
<td>68.8</td>
<td>0.453</td>
</tr>
</tbody>
</table>

Table 5.2 Calculation conditions for Ar propellant (pulsed MPDT).

- Inlet heavy particle temperature (K) $5 \times 10^3$
- Inlet electron temperature (K) $8 \times 10^3$
- Inlet ionization fraction 0.1
- Electron temperature on walls adiabatic
- Sheath voltage (V) 20

Fig. 5.4 Assumed current waveform.

**Temporal Plasma Behavior**

In order to understand the temporal plasma behavior, pressure distributions, ionization fraction distributions, and current paths at the initial phase of a discharge ($t =$
5.1 Pulsed Self-Field MPD Thruster

5, 10 µs), and in steady-state are shown in Figs. 5.5-5.7 for \( \tau = 10 \) µs. The discharge current at \( t = 5, 10 \) µs amounts to 2.8 kA and 4.6 kA respectively. In the present calculation condition, the flow is completely steady-state at the end of discharge (\( t = 250 \) µs). The pressure distributions indicate that a shock is produced in the thruster and propagates towards the downstream region. From the respective result at 5, 10 µs, the shock velocity is estimated as 2.4 km/s. If the gas temperature in the downstream region viewed from the shock front in the reference frame at rest is assumed to be 300 K, the Mach number of the shock is estimated at \( M_s = U_s/a_1 = 7.4 \). However, in terms of the pressure ratio ahead and behind the shock, which is estimated at 10, the \( M_s \) given by the conventional following relation becomes lower value of 3.6 [90]:

\[
M_s = \left( \frac{\gamma - 1}{2} + \frac{\gamma + 1}{2} \frac{p_2}{p_1} \right)^{1/2}.
\]  

(5.5)

This discrepancy is attributed to the fact that the shock is sustained by the intensive Joule heating and the Lorentz force around the cathode, which will play a role like a piston. Thus the conventional equation (5.5) is thought to be inappropriate for the evaluation of \( M_s \) in the present situation. A piston compressing a fluid increases the temperature behind the shock, and then the shock velocity is enhanced in accordance with the raised acoustic velocity.

From Fig. 5.6, it can be seen that the ionization fraction is raised far behind the pressure discontinuity, and the highly ionized region expands radially from the cathode. This suggests that the ionization is mainly caused not by the passing shock but by the Joule heating around the cathode. The ionization fraction amounts to unity in almost whole region in the thruster.

The discharge current path is also affected by the plasma conditions. At \( t = 5 \) µs, the current concentrates around the upstream region in Fig. 5.7 due to high ionization fraction around the inlet. Almost all current paths are confined within the region \( z < 13 \) mm at this moment. After that (\( t > 5 \) µs), the current expands gradually towards the downstream region in accordance with outspreading highly ionized region. Also, the growth of Hall parameter with rising current contributes to the obliquely skewed current profile.
Fig. 5.5 Pressure (Pa), pulsed SFMPDT, Ar, 1.2 g/s, $J_{\text{max}} = 7.2$ kA, $\tau = 10$ μs; (a) 5 μs, (b) 10 μs, (c) Steady State.
5.1 Pulsed Self-Field MPD Thruster

Fig. 5.6 Ionization fraction, pulsed SFMPDT, Ar, 1.2 g/s, $J_{\text{max}} = 7.2$ kA, $\tau = 10$ µs;
(a) 5 µs, (b) 10 µs, (c) Steady-State.
Chapter 5 Attempt toward Low-Power Operation Modes

Fig. 5.7 Current path, pulsed SFMPDT, Ar, 1.2 g/s, $J_{\text{max}} = 7.2$ kA, $\tau = 10$ µs; (a) 5 µs, (b) 10 µs, (c) Steady-State.
5.1 Pulsed Self-Field MPD Thruster

Temporal Variation of Thrust

Thrust of an electric propulsion device is often evaluated by the product of a mass flow rate and an exhaust velocity of a propellant. To obtain a temporal variation of thrust, however, the reaction forces acting on a thruster have to be calculated alternatively, although both evaluations are to provide the same total impulse bit during the discharge. In the present case, thrust is constituted by a thermal component $F_{th}$ acting on the walls, and an electromagnetic force $F_{em}$ acting on the external electrical circuit. The former can be computed by surface integral of the pressure, and the latter is given by the volume integral of the Lorentz force in axial direction:

$$F_{th} = \int_S \rho dS_z = F_c + F_a + F_{iw},$$  \hspace{1cm} (5.6)

$$F_{em} = \int_V (j \times B)_z dV.$$  \hspace{1cm} (5.7)

The pressure component can be subdivided into the contributions from a cathode surface, an anode surface (including the insulator part), and inlet wall ($z = 0, r < 7, r > 11$ mm). Additionally, the contributions of the intrinsic momentum flux at the injection port $F_{in}$ and the friction loss $F_f$ have to be taken into account:

$$F_{in} = \int_{inlet} (\rho u_z^2 + p) dS_z,$$  \hspace{1cm} (5.8)

$$F_f = \int_S \frac{\partial u_z}{\partial \eta} dS_z.$$  \hspace{1cm} (5.9)

Figure 5.8 shows the temporal variation of each thrust component and the discharge current value under the condition of $\tau = 10 \mu$s. The total thrust has a peak around $t = 25 \mu$s, and then thrust is decreased down to 14.5 N. According to each thrust component, this peak is obviously attributed to $F_p$, thus it can be said that the acceleration just after the ignition is dominated by the electrothermal effect. For $t > 100 \mu$s, the thrust becomes constant, which indicates the flowfield reaches a steady state. In the steady state, $F_{em}$ exceeds the other components and saturates at 7.5 N. Although it has been believed that the thrust component of Ar propellant is electromagnetic under the condition $J > J_c$ [91], the calculated result shows that the effect of considerable electrothermal acceleration still remains. For Ar propellant, much higher current than $J_c$ is required to obtain electromagnetic dominant acceleration.
Fig. 5.8 Temporal variation of thrust components, pulsed SFMPDT, Ar, 1.2 g/s, $J_{\text{max}} = 7.2$ kA, $\tau = 10$ µs.

Fig. 5.9 Temporal variation of each thermal thrust component (Anode, Cathode, Inlet wall), pulsed SFMPDT, Ar, 1.2 g/s, $J_{\text{max}} = 7.2$ kA, $\tau = 10$ µs, Solid line is the sum of each component.
The each thermal component described in Fig. 5.9 illustrates that the peak of $F_{th}$ is caused by an increase in pressure on the anode. This increase in pressure is attributed to the shock-propagation. Once the flow becomes steady after 50 µs, the contribution of the inlet wall is the highest, whereas the pressure itself is highest at the cathode tip as shown in Fig. 5.5-(c). In spite of the high pressure on the cathode surface, its contribution seems quite small due to the narrow area of the cathode tip.

**Effects of Discharge Time Constant $\tau$**

The effects of discharge time constant on the plasma behavior and the performance are discussed. In Fig. 5.10, pressure profiles at $r = 10$ mm are described for $\tau = 1$, 10, and 20 µs. When $\tau = 1$ µs, a sharp shock is produced and propagates downstream at a velocity of about 2.4 km/s. The maximum pressure behind the shock ranges from 15-20 kPa. When the discharge time constant is elongated to $\tau = 10$, 20 µs, the pressure in the discharge chamber is decreased down to 5 – 15 kPa, and the pressure profile becomes moderate. When $\tau = 10$ µs, the shock velocity is estimated at 2.4 km/s, and when $\tau = 20$ µs, the shock structure becomes ambiguous.

Specific impulse and thrust efficiency for $T_{on} = 250$ µs are tabulated in Table 5.3. Since the mass shots for all cases are the same because of the same duration time $T_{on}$, the specific impulse depends only on the impulse bit. It can be seen that impulse bit, hence specific impulse, is increased with shortening the discharge time constant. This is obviously due to the difference of the input energy, i.e. the shorter $\tau$ implies higher input energy. What is important is that the thrust efficiency is also slightly improved with shortening $\tau$ due to the increase in the impulse bit despite the increased input energy.
Fig. 5.10 Effect of discharge time constant $\tau$ on pressure profile at $r = 10$ mm, pulsed SFMPDT, Ar, 1.2 g/s, $J_{\text{max}} = 7.2$ kA; (a) $t = 5$ µs, (b) $t = 10$ µs.
5.1 Pulsed Self-Field MPD Thruster

Table 5.3 Performances for $\tau = 1, 10, \text{ and } 20 \, \mu s$, pulsed SFMPDT, Ar, 1.2 g/s, $J_{\text{max}} = 7.2 \, \text{kA}$, $T_{\text{on}} = 250 \, \mu s$. Sheath voltage is assumed to be 20 V.

<table>
<thead>
<tr>
<th>$\tau$ (µs)</th>
<th>$I_b$ (mN·s)</th>
<th>$I_{sp}$ (s)</th>
<th>$\eta$</th>
<th>$\Delta E$ (J)</th>
<th>$\Delta m$ (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.76</td>
<td>853</td>
<td>0.208</td>
<td>75.5</td>
<td>0.453</td>
</tr>
<tr>
<td>10</td>
<td>3.61</td>
<td>819</td>
<td>0.200</td>
<td>72.4</td>
<td>0.453</td>
</tr>
<tr>
<td>20</td>
<td>3.46</td>
<td>783</td>
<td>0.193</td>
<td>68.8</td>
<td>0.453</td>
</tr>
</tbody>
</table>

5.1.6 Effects of Propellant Species

In the previous subsection, it has been demonstrated that both specific impulse and thrust efficiency can be enhanced by shortening the current rising time. This fact suggests that it is preferable to apply higher current, hence higher input energy, to plasma within a constant duration time. However, it is hard to apply higher discharge current, since allowable discharge current is restricted by the critical current. In terms of electrical engineering, there is also limitation on shortening a discharge time constant, because the number of condensers of pulse forming network has to be increased to shorten a discharge time constant. For this reason, a propellant allowing as high critical current as possible under a constant mass flow rate will be preferable, i.e. a propellant of high Alfvén’s critical velocity $u_c$, which is defined in Eq. (1.10), is attractive to increase an input energy.

In the context of a pulsed operation, one of key issues is how to enhance a specific impulse comparable to that of steady operation, since a time-lag of ignition, which is directly linked to waste of propellant, can occur in practical application. To obtain high specific impulse, two guidelines can be considered.

A) Enhancement of allowable discharge current above $J_c$ by optimizing thruster geometry

B) High discharge current about $J_c$ with high $u_c$ propellant

The concept of the segmented anode discussed in Chapter 4 is based on the concept of A). Here, the concept B) is examined. Since $u_c$ is proportional to $M^{-1/2}$ (see Eq.(1.10)), a propellant with light mass is considered to satisfy the present requirement. Generally, high specific impulse can be achieved by using light propellant, especially hydrogenous molecular gases such as $H_2$, $CH_4$, $NH_3$. In addition, other performance parameters also depend on molecular mass. In Fig. 5.11, experimental data under the conditions of $J =$
7.3 kA, \( m = 0.3 \, \text{g/s} \) are plotted [91]. From Fig. 5.11, it can be seen that the properties of helium are similar to hydrogenous molecular gases due to its light mass. The helium’s properties of high ionization energy of 24.59 eV and its light atomic mass enhance \( u_c \) up to 34.5 km/s, whereas \( u_c \) of Ar is 8.7 km/s. Therefore, it will be worthwhile to compare the fundamental plasma behavior of He propellant with those of Ar.

![Fig. 5.11 Gas species comparison of thrust, voltage, and thrust efficiency (7.8 kA, 0.3 g/s, SFMPDT) [91].](image)

**Calculation Conditions**

The input parameters for each case are summarized in Table 5.4. The case2 (Ar, 1.2 g/s, 7.2 kA) is the same condition as that discussed in the previous subsection (see Table 5.1). \( J_{\text{max}} \) is set to the critical current for a given mass flow rate, and the mass flow rate is also varied from 0.4 to 1.2 g/s. The duration time of a discharge (\( T_{\text{on}} \)) is fixed to 250 µs for all cases. Also, the input energy computed with Eq. (5.3) and the massshot are shown in the table. The input energy for He propellant is much higher than that for Ar, partly because the critical current of He is higher than that of Ar. In addition, discharge
voltage for He propellant tends to be higher than that of Ar even under the same discharge current condition, since the back electromotive force and the sheath voltage of He propellant are generally higher than that of Ar.

At the inlet, the heavy particle temperature, the electron temperature, and the ionization fraction for He propellant are assumed to be 1,000 K, 15,000 K, and 0.001 respectively (Table 5.5). The sheath voltage for He is assumed to be 30 V in consideration of its high ionization energy. The calculation conditions for Ar are the same as that used in the previous subsection (Table 5.2).

### Table 5.4 Calculation conditions for comparison of propellant species.

<table>
<thead>
<tr>
<th>Case</th>
<th>Propellant</th>
<th>(\dot{m}) (g/s)</th>
<th>(J_{\text{max}}) (kA)</th>
<th>(\tau) ((\mu)s)</th>
<th>(T_{\text{on}}) ((\mu)s)</th>
<th>(\Delta E) (J)</th>
<th>(\Delta m) (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Ar</td>
<td>1.2</td>
<td>7.2</td>
<td>10</td>
<td>250</td>
<td>72.4</td>
<td>0.453</td>
</tr>
<tr>
<td>4</td>
<td>Ar</td>
<td>0.8</td>
<td>5.9</td>
<td>10</td>
<td>250</td>
<td>55.7</td>
<td>0.302</td>
</tr>
<tr>
<td>5</td>
<td>Ar</td>
<td>0.4</td>
<td>4.1</td>
<td>10</td>
<td>250</td>
<td>34.8</td>
<td>0.155</td>
</tr>
<tr>
<td>6</td>
<td>He</td>
<td>1.2</td>
<td>13.9</td>
<td>10</td>
<td>250</td>
<td>386</td>
<td>0.347</td>
</tr>
<tr>
<td>7</td>
<td>He</td>
<td>0.8</td>
<td>11.3</td>
<td>10</td>
<td>250</td>
<td>267</td>
<td>0.232</td>
</tr>
<tr>
<td>8</td>
<td>He</td>
<td>0.4</td>
<td>8.01</td>
<td>10</td>
<td>250</td>
<td>146</td>
<td>0.117</td>
</tr>
</tbody>
</table>

### Table 5.5 Calculation conditions for He propellant (pulsed MPDT).

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Inlet heavy particle temperature (K)</td>
<td>(1 \times 10^3)</td>
</tr>
<tr>
<td>Inlet electron temperature (K)</td>
<td>(1.5 \times 10^4)</td>
</tr>
<tr>
<td>Inlet ionization fraction</td>
<td>(1 \times 10^{-3})</td>
</tr>
<tr>
<td>Electron temperature on walls</td>
<td>adiabatic</td>
</tr>
<tr>
<td>Sheath voltage (V)</td>
<td>30</td>
</tr>
</tbody>
</table>

**Transient Plasma Behavior**

The pressure distributions at 5, 10 \(\mu\)s, and steady-state for He propellant of 1.2 g/s are described in Fig. 5.12. Comparing the location of shock front at 5 \(\mu\)s and 10 \(\mu\)s, the shock velocity can be estimated as 6.0 km/s. Comparing Fig. 5.12 with Fig. 5.5, however, the shock velocity of He is only 2.5 times as fast as that of Ar. However, considering the mass of a particle, a shock velocity of He should be three times faster than that of Ar for the same temperature. The reason for this difference will be attributed to the difference in temperature in the upstream region. Figures 5.13 and 5.14 show the heavy particle and electron temperatures at the moment when the respective
shock front (discontinuity of pressure profile) is located at $z = 35$ mm ($t = 13.5$ µs for Ar, $t = 6.5$ µs for He). Each figure shows that temperature is highest around the cathode, which implies that main heating of plasma is done not by shock propagation but by Joule heating. Since the specific heat of He per unit mass is higher than that of Ar, the temperature of He during the discharge is less than that of Ar in spite of the higher input energy for He, and the electron temperature as well. The velocity of a shock sustained by some piston-like mechanism is thought to be significantly dependent on the temperature in the upstream region.

The difference of the ionization energy between Ar and He is thought to affect the ionization degree distributions. In Fig. 5.15, temporal evolution of the ionization fraction distributions for He propellant is described. It can be seen that the region with a high ionization fraction for He propellant is limited around the symmetric axis due to its high ionization energy, whereas Ar propellant is fully ionized in the almost whole region within the discharge chamber (Fig. 5.6).

In Fig. 5.16, the temporal evolution of the current paths for He propellant are shown. At each moment, the discharge current for He propellant tends to flow more downstream compared to those for Ar propellant (Fig. 5.7), because a high ionized region, that is, the region with high electrical conductivity is located at the downstream region. Comparing the current paths in steady-state for Ar and He propellant, it can be seen that the current path of He expands much further to the downstream region.
5.1 Pulsed Self-Field MPD Thruster

Fig. 5.12 Pressure (Pa), pulsed SFMPDT, He, 1.2 g/s, $J_{\text{max}} = 13.9$ kA, $\tau = 10$ $\mu$s; (a) 5 $\mu$s, (b) 10 $\mu$s, (c) Steady-State.
Fig. 5.13 Heavy particle temperature at the moment when the each shock front (discontinuity of pressure profile) is located at 35 mm; (a) Ar, $t = 13.5 \mu s$, (b) He, $t = 6.5 \mu s$. 

(a) Ar, $t = 13.5 \mu s$

(b) He, $t = 6.5 \mu s$
Fig. 5.14 Electron temperature at the moment when the each shock front (discontinuity of pressure profile) is located at 35 mm; (a) Ar, $t = 13.5 \, \mu s$, (b) He, $t = 6.5 \mu s$. 

(a) Ar, $t = 13.5 \, \mu s$

(b) He, $t = 6.5 \, \mu s$
Fig. 5.15 Ionization fraction, pulsed SFMPDT, He, 1.2 g/s, $J_{\text{max}} = 13.9$ kA, $\tau = 10$ µs; (a) 5 µs, (b) 10 µs, (c) Steady-State.
Fig. 5.16 Current path, pulsed SFMPDT, He, 1.2 g/s, $J_{max} = 13.9$ kA, $\tau = 10$ $\mu$s; (a) 5 $\mu$s, (b) 10 $\mu$s, (c) Steady-State.
Temporal Variation of Thrust

The temporal variations of thrust for He propellant for $m = 1.2$ g/s (case6) are plotted in Fig. 5.17. Different from the thrust curve for Ar propellant shown in Fig. 5.8, the total thrust has no peak during the discharge. The electromagnetic thrust $F_{em}$ of He amounts to as high as 32 N, whereas $F_{em}$ of Ar is only 7.5 N. The reason for this is obviously attributed to the difference of the maximum current, namely the critical current, of each propellant. In the steady state, the ratio of the electromagnetic thrust to the total thrust is 51% for Ar and 89% for He. In terms of impulse bit within 250 µs, the ratio of the electromagnetic impulse bit to the total impulse bit is 49% for Ar and 85% for He. This result suggests that He propellant is more suitable for the electromagnetic acceleration than Ar, which is obvious as seen from the definition of the critical current Eq. (1.10), i.e. the higher the critical current is, the larger current can be applied for a constant mass flow rate.

Although the magnitude of electromagnetic thrust is quite different between the both propellants, the electrothermal components are quite close as can be seen from Figs. 5.8 and 5.17. However, individual contributions of each part (anode, cathode, inlet wall ($z = 0, \ r < 7, \ r > 11 \ mm$)) to thermal thrust are different between the both propellants. Figure 5.18 shows the temporal variation of the individual pressure thrust for He propellant. While the contribution of the inlet wall amounts to 2.4 N in the case of Ar (Fig. 5.9), the counterpart of He is no more than 1.2 N (for $t > 40 \ µs$). This is because Joule heating for He mainly occurs in the downstream region compared to Ar. Also, the thermal thrust by the cathode exceeds the contribution from the anode in the case of He, while the thermal thrust by the anode of Ar is much higher than that by the cathode as shown in Fig. 5.9. This is due to the fact that, high pressure region of He propellant is mainly distributed around cathode tip, and is confined by the pinch force produced by the highly skewed current mentioned above.
5.1 Pulsed Self-Field MPD Thruster

Fig. 5.17 Temporal variation of thrust components. He, 1.2 g/s, $J_{max} = 13.9$ kA, $\tau = 10$ µs.

Fig. 5.18 Temporal variation of each pressure component (Anode, Cathode, Inlet wall). Solid line is the sum of each component. He, 1.2 g/s, $J_{max} = 13.9$ kA, $\tau = 10$ µs.
Performance Evaluation

Relationships between thrust efficiency and specific impulse evaluated with Eq. (5.4) are summarized in Fig. 5.19. As expected, specific impulse for He becomes much higher than that for Ar. For all mass flow rates, the specific impulses show quite close values about 800 s for Ar and about 2,500 s for He. The high specific impulse of He propellant is attributed to its high ratio of the electromagnetic thrust component in the thrust. If the electromagnetic thrust is dominant, the attainable specific impulse will be close to $u_c/g$ under the condition of $J = J_c$, where $u_c$ is the Alfvén’s critical velocity:

$$I_{sp} = \frac{F}{\dot{m}g} \approx \frac{b J_c^2}{\dot{m} g b} = \frac{b u_c}{g b} = \frac{u_c}{g}. \quad (5.10)$$

The specific impulse of He obtained with a discharge duration time of 250 µs, however, is much less than its $u_c/g$ (= 3,500 s). Since electromagnetic thrust is dominant at any time during the discharge, specific impulse is considerably affected by the current rising phase ($0 < t < 50 \mu s$), i.e. the $I_{sp}$ much less than $u_c/g$ is mainly attributed to the low electromagnetic thrust during the current rising phase. However, even if the discharge duration time $T_{on}$ is further elongated, $I_{sp}$ of He is limited to about 3,100 s as will be shown later. On the other hand, the specific impulse of Ar achieves the value comparable to its $u_c/g$ (= 870 s) with a discharge duration time of 250 µs despite the fact that the electromagnetic component of thrust is about 50 % as mentioned above. This is because the thrust is constituted by the hybrid of the electromagnetic and the electrothermal components. The total thrust is complemented by the electrothermal component during the current rising phase, thus it is inferred that specific impulse of Ar can exceed $u_c/g$ of Ar with the aid of thermal thrust. Actually, if the discharge duration time is further extended, $I_{sp}$ of Ar amounts to about as high as 1,200 s. This indicates that the thrust contains considerable contributions of pressure component.

The thrust efficiency for He propellant is basically higher than that of Ar, and is increased with mass flow rate, even though the momentum coupling coefficients of He represented by the dashed lines in Fig. 5.19 are inferior to that of Ar. These tendencies accord with the experimental results obtained by Yoshikawa [23]. An increase in thrust efficiency with an increasing mass flow rate indicates the enhancement of momentum coupling coefficients.
5.1 Pulsed Self-Field MPD Thruster

Fig. 5.19 Relation between thrust efficiency and specific impulse for pulsed MPDT (Discharge duration time is 250 µs). Sheath voltage is assumed to be 20 V for Ar and 30 V for He. Dashed lines represent momentum coupling coefficients of 50, 44, 35, 23, 21, and 19 mN/kW.

Effects of Discharge Duration Time

The performances discussed above are evaluated with the results under the condition of $T_{on} = 250 \text{ µs}$. Since the performance parameters such as the impulse bit are integrated value with regard to time, changing the discharge time length $T_{on}$ can alter the results. It is easily anticipated that the performances will approach those of steady-state operation as the discharge time length is elongated. Here, relation between the performances and the discharge time length is argued.

In Fig. 5.20, the relationship between the specific impulse $I_{sp}$ and the discharge time length $T_{on}$ is shown. It can be seen that $I_{sp}$ is gradually increased with the discharge time length. For instance, in the case of He 1.2g/s, $I_{sp}$ is improved from 2,000 to 2,750 s by extending the $T_{on}$ from 100 to 600 µs. Similar tendency can also be found in thrust efficiency $\eta$ as shown in Fig. 5.21. It can be seen that the specific impulse and the thrust efficiency tend to saturate around $t = 400 – 600 \text{ µs}$. Comparing the performances of $T_{on} = 600 \text{ µs}$ with those of steady operation (Table 5.6), it can be seen that performances comparable to that of steady operation can be achieved with $T_{on} = 600 \text{ µs}$. In terms of
the ratio of performances for $T_{on} = 600 \, \mu s$ to those of steady operation, He propellant seems superior to Ar. Since extending discharge duration time entails to enlarge a capacitor of a power source, tradeoff has to be done in designing a pulsed MPDT system.

Fig. 5.20 Relation between specific impulse and discharge duration time.

Fig. 5.21 Relation between thrust efficiency and discharge duration time. Sheath voltage is assumed to be 20 V for Ar and 30 V for He.
Table 5.6 Comparison of performances for $T_{on}=600 \mu$s with those of steady operation. Sheath voltage is assumed to be 20 V for Ar and 30 V for He.

<table>
<thead>
<tr>
<th>Propellant</th>
<th>$\dot{m}$ (g/s)</th>
<th>$I_{sp}$ (s) (600µs)</th>
<th>$\eta$ (600µs)</th>
<th>$I_{sp}$ (s) (steady)</th>
<th>$\eta$ (steady)</th>
<th>$I_{sp}(600µs)$</th>
<th>$\eta(600µs)$</th>
<th>$I_{sp}(steady)$</th>
<th>$\eta(steady)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>He</td>
<td>1.2</td>
<td>2750</td>
<td>0.31</td>
<td>3097</td>
<td>0.34</td>
<td>0.89</td>
<td>0.92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ar</td>
<td>1.2</td>
<td>1000</td>
<td>0.24</td>
<td>1220</td>
<td>0.28</td>
<td>0.82</td>
<td>0.85</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5.1.7 Summary

The temporal plasma behavior and performances of a pulsed SFMPDT are numerically elaborated. The current waveform is assumed by a simple function of a maximum current $J_{max}$ and a discharge time constant $\tau$. Discharge duration time $T_{on}$ is fixed to 250 µs. When $\tau = 10$ µs, a shock wave is produced and propagates toward a downstream region, which is sustained by intensive Joule heating and Lorentz force around the cathode. However, the shock profile becomes ambiguous for extended $\tau$.

Under the condition of a constant mass flow rate of 1.2 g/s (Ar propellant) and $J_{max} = J_c$, specific impulse and thrust efficiency are improved from 783 s and 19% to 853 s and 21% by shortening the discharge time constant $\tau$ from 20 to 1 µs.

To obtain higher specific impulse, which is the key issue for a pulsed operation, helium propellant is employed to compare its properties with that of argon. Owing to its high ionization energy and light atomic weight, allowable maximum current (critical current) becomes much higher than that for Ar. This makes electromagnetic acceleration dominant (89%) for He propellant, whereas the electrothermal component comparable to the electromagnetic thrust still remains for Ar propellant. The high electromagnetic thrust causes specific impulse of about 2,500 s for He with the discharge duration time of 250 µs.

The effects of the discharge duration time ($T_{on}$) on the performances are also examined. It is specified that specific impulse and thrust efficiency are improved with extending $T_{on}$. Specific impulse and thrust efficiency of He approaches those of steady state faster than Ar.
5.2 Applied-Field MPD Thrusters

When an external magnetic field is applied to an MPDT, additional acceleration mechanisms are added to the self-field acceleration as illustrated in Subsec. 1.2.5. Since some acceleration mechanisms coexist in an AFMPDT, clarification of the thrust components and their energy conversion processes will be of importance. This section aims to simulate the flowfields of several tens of kW level AFMPDT to examine thrust production mechanisms. The strength of applied magnetic field is varied from 0.08 to 0.25 T under a constant discharge current of 1 kA, and a constant Ar mass flow rate of 0.1 g/s. Two magnetic field configurations are employed to examine the effect of diverging angle of the magnetic field on plasma flowfields.

5.2.1 Modeling of Applied-Field MPD Thruster

Vector Potential

In the flowfields of SFMPDTs, only the azimuthal component of the induction equation is necessary to describe an arc discharge between a cathode and an anode. On the other hand, in AFMPDTs, the axial and radial magnetic field components are additionally needed to include the effects of the applied magnetic field and corresponding induced azimuthal current. To incorporate these influences, the vector potential is employed for the axial and radial components of the magnetic field, while the azimuthal magnetic field induced by the discharge current is computed via the induction equation:

\[
\begin{pmatrix}
B_r \\
B_\theta \\
B_z
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \text{rot} \begin{pmatrix} A_\theta \\ 0 \\ 0 \end{pmatrix}.
\] (5.11)

Under the assumption of the azimuthal uniformity (\(\partial / \partial \theta = 0\)), \(\text{div} B = 0\) and \(\text{div} A = 0\) are mathematically ensured with this formulation. Since only the flowfields in steady-state are interested, the steady form of the vector potential equation is solved every several time steps:

\[
\frac{1}{\mu_0 \sigma} \left( \nabla^2 A_\theta - \frac{A_\theta}{r^2} \right) + (u_r B_r - u_z B_z) - \frac{1}{en} (j_r B_r - j_z B_z) = 0.
\] (5.12)

In Eq. (5.12), \(B_r\) and \(B_z\) include not only the original applied magnetic field \(B_r^{ap}, B_z^{ap}\) but
also the induced magnetic field which is given by the derivatives of the vector potential as shown below:

\[
B_r = B_r^{ap} - \frac{\partial A_\theta}{\partial z}, \quad B_z = B_z^{ap} + \frac{1}{r} \frac{\partial r A_\theta}{\partial r}.
\] (5.13)

As for the current density in the above equations, the radial and axial components are computed with the Ampère’s law, and the azimuthal component is calculated from the Ohm’s law with the assumption of azimuthal uniformity:

\[
j_r = -\frac{1}{\mu_0 r} \frac{\partial \psi}{\partial z},
\] (5.14)

\[
j_\theta = \sigma \left( u_z B_r - u_r B_z - \frac{1}{en_e} (j_z B_r - j_r B_z) \right),
\] (5.15)

\[
j_z = \frac{1}{\mu_0 r} \frac{\partial \psi}{\partial r}.
\] (5.16)

The original applied magnetic field in Eq. (5.13) produced by an external coil with a coil-current \(J_{coil}\) has to be computed at each point in advance. In the axisymmetric field, the vector potential at an arbitrary point \((r_j, z_j)\) produced by each coil-current loop \(i\) at \((r_{c,i}, z_{c,i})\) is given by (See Fig. 5.22 for notation) [44]

\[
A_{\theta,j}^{ap} = \sum_i \frac{\mu_0 J_{coil}}{2\pi} \sqrt{\frac{r_{c,i}}{r_j}} \left\{ \frac{2 - k_i^2}{k_i} K(k_i) - 2E(k_i) \right\},
\] (5.17)

with the definitions of

\[
k_i = \sqrt{\frac{4r_j r_{c,i}}{(r_j + r_{c,i})^2 + (z_j - z_{c,i})^2}},
\] (5.18)

\[
K(k_i) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k_i \sin^2 \theta}}, \quad E(k_i) = \int_0^{\pi/2} \sqrt{1-k_i^2 \sin^2 \theta} \, d\theta.
\] (5.19)

The rotation of Eq. (5.17) provides the radial and axial magnetic field at \((r_j, z_j)\):

\[
B_{r,j}^{ap} = \sum_i \frac{\mu_0 J_{coil}}{2\pi \sqrt{(r_j + r_{c,i})^2 + (z_j - z_{c,i})^2}} \left\{ -K(k_i) + \frac{r_{c,i}^2 + r_j^2 + (z_j - z_{c,i})^2}{(r_{c,i} - r_j)^2 + (z_j - z_{c,i})^2} E(k_i) \right\},
\] (5.20)

\[
B_{z,j}^{ap} = \sum_i \frac{\mu_0 J_{coil}}{2\pi \sqrt{(r_j + r_{c,i})^2 + (z_j - z_{c,i})^2}} \left\{ K(k_i) + \frac{r_{c,i}^2 - r_j^2 - (z_j - z_{c,i})^2}{(r_{c,i} - r_j)^2 + (z_j - z_{c,i})^2} E(k_i) \right\}.
\] (5.21)
Electron pressure gradient

In the analysis of AFMPDT, the effect of electron pressure gradient is taken into account in the generalized Ohm’s law:

\[
j = \sigma \left( E + u \times B - \frac{1}{en_e} j \times B + \frac{1}{en_e} \nabla p_e \right). \tag{5.22}\]

The electron pressure gradient can be interpreted as a source of diamagnetic current, which can contribute to the azimuthal induced current within an AFMPDT [92]. In this case, Eqs. (2.11) and (2.12) have to be altered as follows:

\[
\frac{\partial}{\partial t} (U_e + U_i) + \nabla \cdot [(U_e + U_i) u] = -p_e \nabla \cdot u + \frac{j^2}{\sigma} + \nabla \cdot (\lambda_e \nabla T_e) + \frac{5k}{2e} j \cdot \nabla T_e - \frac{1}{en_e} j \cdot \nabla p_e - \delta E, \tag{5.23}\]

\[
\frac{\partial B}{\partial t} - \nabla \times (u \times B) = -\nabla \left[ \frac{1}{\mu_0 \sigma} \nabla \times B + \frac{1}{\mu_0 en_e} (\nabla \times B) \times B - \frac{1}{en_e} \nabla p_e \right]. \tag{5.24}\]

Anomalous Resistivity

Generally, an AFMPDT with several tens of kW is operated with relatively lower mass flow rate than that of SFMPDT. It has been pointed out that the classical model defined by Eq. (2.20) is insufficient to evaluate the effective conductivity in an AFMPDT flowfield because of the properties of low density and strong magnetic field.
5.2 Applied-Field MPD Thrusters

It is known that the current can drive microinstabilities which can substantially enhance dissipation in plasma, and then the effective conductivity is affected by the effect [58]. As a model for the anomalous resistivity, the Chodura’s model, which assumes that the lower-hybrid drift microinstability is significant, is used in this study [43]:

$$\eta_{an} = \frac{0.7 m_e}{e \sqrt{\varepsilon_0}} \frac{M'}{\rho} \left( 1 - e^{-u_{e,d} \ln_{10}} \right) \left( 1 + 0.3 \frac{u_{e,d}}{u_{i,ac}} \sqrt{\frac{|B|^2}{|B|^2 + C' \rho / M'}} \right). \quad (5.25)$$

Here, $C'$ is a constant equal to $6.1544 \times 10^7$. The $u_{e,d}$ and $u_{i,ac}$ denote an electron drift velocity and an ion acoustic velocity respectively:

$$u_{e,d} = \frac{|j|}{en_e}, \quad u_{i,ac} = \sqrt{\frac{jk(T_n + \alpha T_e)}{m_n}}. \quad (5.26)$$

The electrical conductivity is defined by the inverse of the sum of the classical the anomalous resistivity:

$$\frac{1}{\sigma} = \frac{1}{\sigma_{\text{classic}}} + \eta_{an}. \quad (5.27)$$

**Boundary Condition for Vector Potential Formulation**

Based on the Biot-Savart law, the induced vector potential $A_{\theta,\text{ind}}^{\text{ind}}$, and radial, axial induced magnetic fields $B_r^{\text{ind}}$, $B_z^{\text{ind}}$ at boundaries $(r_{b,i}, z_{b,i})$ are influenced by the induced azimuthal current in the whole computational region. Therefore, they have to be specified at boundaries in the iterative procedure of Eq. (5.12). This can be done with Eqs. (5.17), (5.20), and (5.21) by replacing $J_{\text{coil}}$ with $j_{\theta,j} S_j$, where $j_{\theta,j}$ and $S_j$ are an azimuthal current density at a cell $j$ and an area of a cell $j$ respectively (See Fig. 5.22 for notation):

$$A_{\theta,b,i}^{\text{ind}} = \sum_{j=1}^{\mu} \frac{\mu_0 j_{\theta,j} S_j}{2\pi} \frac{1}{r_{b,i}} \left\{ \frac{2 - k_j^2}{k_j} \right\}, \quad (5.28)$$

$$B_r^{\text{ind}} = \sum_{j=1}^{\mu} \frac{\mu_0 j_{\theta,j} S_j}{2\pi} \sqrt{(r_{b,i} + r_j)^2 + (z_{b,i} - z_j)^2} \left\{ -K(k_j) \left[ \frac{r_j + r_{b,i}^2 + (z_{b,i} - z_j)^2}{(r_j - r_{b,i})^2 + (z_{b,i} - z_j)^2} E(k_j) \right] \right\}, \quad (5.29)$$

$$B_z^{\text{ind}} = \sum_{j=1}^{\mu} \frac{\mu_0 j_{\theta,j} S_j}{2\pi} \sqrt{(r_{b,i} + r_j)^2 + (z_{b,i} - z_j)^2} \left\{ K(k_j) \left[ \frac{r_j^2 - r_{b,i}^2 - (z_{b,i} - z_j)^2}{(r_j - r_{b,i})^2 + (z_{b,i} - z_j)^2} E(k_j) \right] \right\}, \quad (5.30)$$
where
\[
k_j = \frac{4 r_{b_j} r_j}{\sqrt{(r_{b_j} + r_j)^2 + (z_{b_j} - z_j)^2}}.
\]

According to Eq. (5.25), the electrical conductivity depends on the current density \( j \) via the electron drift velocity \( u_{e,d} \), which means that the azimuthal current and the electrical conductivity are correlated with each other. Consequently, the evaluation of the electrical conductivity has to be incorporated in the iterative solver for Eq. (5.12).

### 5.2.2 Thruster Configuration and Applied Magnetic Field

In Fig. 5.23, coaxial thruster geometry and computational grid are illustrated. As a thruster, a flared type anode is employed, and a short cathode is adopted. The propellant is fed from a port located at the center of the inlet wall, i.e. \( z = 0, r = 7 – 11 \text{ mm} \). To examine the effects of the applied magnetic field on the flowfields well downstream of the thruster exit as well as within the thruster, the computational grid is extended up to about 300 mm downstream. The anode is radially extended at the thruster exit up to 50 mm.

The applied magnetic field is produced by an external coil surrounding the anode. Here, two types of external coil are adopted to examine the effect of magnetic field configuration on the flowfields. The parameters of the coil (axial location, radial location, number of windings and layers, coil current, maximum magnetic field \( B_{\text{max}} \)) are shown in Table 5.7. By varying the radial location of the coil, the magnetic field configuration is altered. The respective coils of \( r = 4-14 \text{ mm} \) and \( r = 10-20 \text{ mm} \) are denoted as C1 and C2 in this thesis. In Fig. 5.24, the applied magnetic field distributions for \( B_{\text{max}} = 0.25 \text{ T} \) with C1 and \( B_{\text{max}} = 0.16 \text{ T} \) with C2 are described. Since the radial position of the C2 is outer than that of C1, the magnetic field of C2 less diminishes in the plume region than that of C1 in spite of lower \( B_{\text{max}} \). The magnetic field lines of C1 are almost parallel to the solid nozzle at the thruster exit, whereas those of C2 are less diverging.
5.2 Applied-Field MPD Thrusters

Fig. 5.23 Thruster geometry and computational grid for AFMPDT. The propellant is fed from a port located at the center of the inlet wall, i.e. $z = 0$, $r = 7 - 11$ mm.

Table 5.7 Parameters for external coils.

<table>
<thead>
<tr>
<th>Coil</th>
<th>Axial (cm)</th>
<th>Radial (cm)</th>
<th>Windings</th>
<th>Layers</th>
<th>Coil Current (A)</th>
<th>$B_{\text{max}}$ (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>-6 ~ 4</td>
<td>4 ~ 14</td>
<td>40</td>
<td>40</td>
<td>12.5, 18.75, 25</td>
<td>0.125, 0.19, 0.25</td>
</tr>
<tr>
<td>C2</td>
<td>-6 ~ 4</td>
<td>10 ~ 20</td>
<td>40</td>
<td>40</td>
<td>12.5, 18.75, 25</td>
<td>0.08, 0.12, 0.16</td>
</tr>
</tbody>
</table>

Fig. 5.24 Magnetic field configuration for analyses of AFMPDT; top: $B_{\text{max}}$=0.25 T with C1, bottom: $B_{\text{max}}$=0.16 T with C2. Parameters of the coils (C1,C2) are shown in Table 5.7.
5.2.3 Calculation Conditions

At the inlet, the mass flow rate is kept to 0.1 g/s, and the heavy particle temperature, the electron temperature, and the ionization fraction are set to $10^3$ K, $10^4$ K, and 0.1 respectively. The total discharge current is kept to 1 kA. Under the present conditions, the strength of the azimuthal induced magnetic field producing self-field acceleration is less than 0.05 T at most, thus the applied-field is dominant in the flowfield. The electron temperature is kept less than 2 eV on the walls.

<table>
<thead>
<tr>
<th>Table 5.8 Calculation conditions for analyses of AFMPDT flowfields.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Propellant</td>
</tr>
<tr>
<td>Mass flow rate (g/s)</td>
</tr>
<tr>
<td>Discharge current (kA)</td>
</tr>
<tr>
<td>Inlet heavy particle temperature (K)</td>
</tr>
<tr>
<td>Inlet electron temperature (K)</td>
</tr>
<tr>
<td>Inlet ionization fraction</td>
</tr>
<tr>
<td>Electron temperature on walls</td>
</tr>
</tbody>
</table>

5.2.4 Flowfields of Applied-Field MPD Thruster

Discharge Current Distribution

To see the effect of the strength of the applied magnetic field on the discharge, the discharge current paths, which means the current contour lines in z-r plane, for $B_{\text{max}} = 0.125, 0.25$ T with C1, and $B_{\text{max}} = 0.16$ T with C2 are depicted in Fig. 5.25. The labeled values denote the ratio of the discharge current flowing in the upstream region viewed from the contour line to the total discharge current. For comparison, the current path without applied magnetic field, namely current path of SFMPDT, is also shown in Fig. 5.26. In the case of SFMPDT, almost all of the discharge current is confined within the thruster ($z < 55$ mm). On the other hand, in the case of AFMPDT, although most of the discharge current paths are basically confined within the thruster, a fraction of the discharge current is expanded toward the downstream region. Comparing the Fig. 5.25-(a) with (b), it can be seen that the current expansion is encouraged by increasing the applied magnetic field strength. The reason for this is due to the Hall effect, i.e. the effective electrical conductivity along the magnetic field lines is larger than that perpendicular to the magnetic field by a factor of $1/(1+\beta_e^2)$. As far as the plume region
is concerned, since the magnetic field strength of the configuration \( B_{\text{max}} = 0.16 \text{ T} \) with C2 is higher than that of \( B_{\text{max}} = 0.25 \text{ T} \) with C1 as shown in Fig. 5.24, the current path of Fig. 5.25-(c) is further expanded than that of Fig. 5.25-(b).

Comparing the Fig. 5.25 with the Fig. 5.26, the discharge current appears to shift upstream within the thruster by applying the magnetic field application. In Fig. 5.27, the discharge current density in \( z-r \) plane \((\sqrt{j^2_r + j^2_z})\) on cathode surface is plotted. Certainly, it can be seen that the current density around \( z = 2-4 \text{ mm} \) is increased by the magnetic field application. This can be interpreted as follows. There is a swirl flow around the cathode as shown in Fig. 5.28, which is driven by the azimuthal Lorentz force \(-j_z B_z\). The \( u_\theta \) rapidly increases around the cathode, and amounts to about 10 km/s at \( z \approx 6 \text{ mm} \). Since this azimuthal velocity \( u_\theta \) causes the back electromotive force \( u_\theta B_z \) toward the anode, the discharge current inlines to evade the region with high \( u_\theta \). Actually the back electromotive force \( u_\theta B_z \) amounts to \( O(10^3) \text{ V/m} \) at \( z \approx 6 \text{ mm} \), which can be comparable to the static electric field. However, when the magnetic field is increased from 0.125 to 0.25 T, the concentration around \( z = 2-3 \text{ mm} \) is reduced and the current density around cathode tip begins to recover. This is due to the Hall effect as mentioned above, i.e. the Hall effect makes the current path skewed toward the downstream region, which requisitely causes an increase in current density at the cathode tip. This tendency of the current shift toward the upstream region was actually confirmed experimentally by Tahara [38], where a similar electrode configuration to the present case was employed. On the other hand, the increase in current density at the cathode tip with magnetic field strength was also observed by Kimura [93], where a converging-diverging anode was used. Therefore, the current density on cathode surface seems to strongly depend on the electrode configuration and working conditions.

**Velocity Distributions**

As mentioned above, the radial current and the axial magnetic field produce an azimuthal Lorentz force, and then a swirl flow is induced mainly around the cathode. Figure 5.28 shows the azimuthal velocity distributions for \( B_{\text{max}} = 0.125, 0.25 \text{ T} \) with C1, and \( B_{\text{max}} = 0.16 \text{ T} \) with C2. The current density and the axial magnetic field strength are highest around the cathode, hence the azimuthal Lorentz force as well. The
interaction parameter $s$ defined by the ratio of azimuthal Lorenz force to the inertial force amounts to as high as 10 at the inlet (characteristic values: $j_r \approx 10^6$ A/m$^2$, $B_z \approx 0.1$ T, $\rho_c \approx 10^{-4}$ kg/m$^3$, $u_c \approx 10^3$ m/s, $L_c \approx 10^{-2}$ m):

$$s = \frac{j_r B_z L_c}{\rho_c u_c^2} \approx 10.$$ (5.31)

Comparing the results of $B_{\text{max}} = 0.125, 0.25$ T with C1, the maximum azimuthal velocity seems to saturate at about 10 km/s. This tendency accords with the simulation results obtained by Mikellides, where it was suggested that the azimuthal velocity is limited by the viscous force against the azimuthal Lorentz force [43]. The azimuthal velocity is rapidly decreased in the flared part. It diminishes to about 3 – 4 km/s at the thruster exit, and gradually approaches to zero in the plume region. It should be noted that there appears a region with a negative azimuthal velocity in the plume region near the insulator. This is because the direction of the azimuthal Lorentz force $-j_r B_z$ inverts due to the radial expansion of the discharge current path in this region as shown in Fig. 5.25 (See also Fig. 5.29). From Fig. 5.28-(b) and (c), it can be seen that this tendency becomes noticeable as the diverging angle of the applied magnetic field is reduced.
5.2 Applied-Field MPD Thrusters

(a) $B_{\text{max}} = 0.125$ T (C1)

(b) $B_{\text{max}} = 0.25$ T (C1)

(c) $B_{\text{max}} = 0.16$ T (C2)

Fig. 5.25 Current path with applied magnetic field, Ar, 0.1 g/s, $J = 1$ kA; (a) $B_{\text{max}}=0.125$ T with C1, (b) $B_{\text{max}}=0.25$ T with C1, (c) $B_{\text{max}}=0.16$ T with C2. The labeled values denote the ratio of the discharge current flowing in the upstream region viewed from the contour line to the total discharge current. Parameters of the coils (C1,C2) are shown in Table 5.7.
Fig. 5.26 Current path without applied-field (self-field), Ar, 0.1 g/s, $J = 1\text{kA}$.

Fig. 5.27 Discharge current density in $z$-$r$ plane ($\sqrt{j_r^2 + j_z^2}$) on cathode surface for self-field, $B_{\text{max}}=0.125\text{ T}$ with C1, and $B_{\text{max}}=0.25\text{ T}$ with C1, Ar, 0.1 g/s, $J = 1\text{kA}$. The cathode tip is located at $z = 13\text{ mm}$. Parameters of the coils (C1,C2) are shown in Table 5.7.
5.2 Applied-Field MPD Thrusters

Fig. 5.28 Azimuthal velocity (km/s), Ar, 0.1 g/s, $J = 1$ kA; (a) $B_{\text{max}} = 0.125$ T with C1, (b) $B_{\text{max}} = 0.25$ T with C1, (c) $B_{\text{max}} = 0.16$ T with C2. Parameters of the coils (C1,C2) are shown in Table 5.7.
In Fig. 5.30, the velocity distributions in z-r plane are described. From Fig. 5.30-(a) and (b), it can be said that the attainable velocity is increased with the strength of applied magnetic field. For $B_{\text{max}} = 0.25$ T with C1, the maximum velocity amounts to above 10 km/s at the end of the computational region. However, it should be noted that, since the discharge voltage is increased with applied magnetic field as will be shown in the next subsection, input energy to the propellant for $B_{\text{max}} = 0.25$ T with C1 is higher than that for $B_{\text{max}} = 0.125$ T with C1. Regarding the case for $B_{\text{max}} = 0.16$ T with C2, the attainable velocity is limited to about 9 km/s, and the speed of acceleration seems less than that for $B_{\text{max}} = 0.125$ T with C1 despite the higher $B_{\text{max}}$. This is due to the difference in the effective cross section of the magnetic nozzle. The difference in these two cases can tell us that the plasma interacts with the magnetic field very well in the plume region, although the streamlines does not follow the magnetic field lines completely. At the thruster exit ($z = 55$ mm), the axial velocity is almost the same about 7.0 km/s in all cases, thus it can be said that the divergence angle of the magnetic field have a significant influence on the axial velocity in the plume region.
Fig. 5.30 Velocity in z-r plane (km/s), Ar, 0.1 g/s, $J = 1$ kA; (a) $B_{\text{max}}=0.125$ T with C1, (b) $B_{\text{max}}=0.25$ T with C1, (c) $B_{\text{max}}=0.16$ T with C2. Parameters of the coils (C1,C2) are shown in Table 5.7.
5.2.5 Thrust Production and Performance

It has been commonly believed that the applied magnetic field provides additional acceleration mechanisms aside from the self-field acceleration; they are called as Hall acceleration, and swirl acceleration. However it has to be noted that the thrust component of Hall acceleration and swirl acceleration cannot be subdivided specifically, because the conversion of the azimuthal momentum into the axial momentum through the effect of magnetic nozzle entails the axial Lorentz force produced by the azimuthal current \( j_\theta \) and the radial magnetic field \( B_r \), which seems equivalent to the Hall acceleration [35]. Hence the Hall acceleration can be interpreted as a steady-state expansion of a rotating fluid confined to a magnetic flux tube. In addition, from a different viewpoint, the rotating motion will result in compression of the fluid toward the surface of a solid nozzle. This effect is considered to increase the pressure on the solid nozzle, which certainly corresponds to enhancement of the aerodynamic acceleration. Therefore, when we discuss contribution of some acceleration mechanisms, specific classification is difficult, i.e. the each thrust component cannot be isolated. This is one of the most detrimental factors to confuse a most efficient operation condition of an AFMPDT, which is still subject of considerable debate.

On the other hand, the energy components for the each acceleration mechanism can be easily classified, e.g. the energy per unit volume, unit time for Hall acceleration is \( -j_\theta B_r u_z \), and for swirl acceleration is \( -j_r B_z u_\theta \). Based on this classification, a thrust formula was derived by Sasoh under the assumption that the work done by electromagnetic forces is completely converted into the axial kinetic energy of the propellant [94], where aerodynamic acceleration was ignored, and \( j_r/j_\theta \ll 1 \), and \( B_r/B_z \ll 1 \) was assumed. The resulting formula is a nonlinear function of the each thrust components.

In this study, the total thrust \( F \) is classified based on parts on which a reaction forces of a thrust act; thermal (aerodynamic) thrust \( F_{\text{therm}} \), self-field thrust \( F_{\text{self}} \), a thrust acting on a external coil \( F_{\text{Hall}} \), and friction loss \( F_f \). Strictly speaking, the propellant has an intrinsic momentum flux at the inlet \( F_{in} \), thus this has to be involved:

\[
F = F_{\text{therm}} + F_{\text{self}} + F_{\text{Hall}} + F_{in} + F_f.
\] (5.32)

The aerodynamic thrust is equivalent to an axial component of the pressure acting on the walls:
5.2 Applied-Field MPD Thrusters

\[ F_{\text{therm}} = \int_{\text{wall}} p dS_z. \]  

(5.33)

The aerodynamic thrust defied here contains the pumping force which is result from the pressure on a cathode increased by pinch force.

The self-field thrust actually acts on the external electrical circuit for the discharge, which is based on the principle that a self-force has to vanish in a system under consideration. This component can be defined as a volume integral of the blowing force with regard to the self-field:

\[ F_{\text{self}} = \int_{V} j_{\theta} B_{\theta} dV. \]  

(5.34)

The thrust acting on an external coil is yielded by the interaction between the induced magnetic field produced by the induced azimuthal current and the coil current. In other words, repulsive force between a coil current and the azimuthally induced current contributes to thrust production. The azimuthal current \( j_{\theta} \) is basically induced by the Hall current and the diamagnetic current. This component can be evaluated by a volume integral of the axial Lorentz force \(-j_{\theta} B_{r}\):

\[ F_{\text{Hall}} = \int_{V} -j_{\theta} B_{r} dV. \]  

(5.35)

Also, the friction loss \( F_{f} \), which has usually a negative value, and the momentum flux at the inlet \( F_{in} \) can be directly computed from the calculated results.

The dependences of the total thrust and the discharge voltage on the applied magnetic field are plotted in Fig. 5.31. Note that the discharge voltage does not include a sheath voltage \( V_{sh} \). It can be seen that the thrust and the discharge voltage tend to linearly increase with the strength of applied magnetic field regardless of the magnetic field configuration. The thrust and discharge voltage amount to 1.34 N and 81.2 V respectively for \( B_{\text{max}} = 0.25 \) T with C1, then the specific impulse and the thrust efficiency is 1,370 s and 8.9% \((V_s = 20 \text{ V})\) respectively, which are highest performances obtained under the present conditions (Table 5.9). Compared to the experimental data under the conditions of \( J = 1 \text{kA}, 0.1 \text{ g/s} \) of Ar propellant, \( P \approx 100 \text{ kW} \) conducted by Myers shown in Fig. 5.32, the performances obtained from this calculation \((I_{sp}=1,370 \text{ s}, \eta= 8.9\%)\) seem relatively low, although the present thruster geometry is much smaller than Myers’s. Scale merit concerning thermal energy loss may enhance the thrust efficiency of Myer’s thruster.
Fig. 5.31 Total thrust and discharge voltage, Ar, 0.1 g/s, J = 1kA. Sheath voltage is not included in the voltage. Parameters of the coils (C1,C2) are shown in Table 5.7.

Table 5.9 Performances, Ar, 0.1 g/s, J = 1 kA. Sheath voltage is assumed to be 20 V. Parameters of the coils (C1,C2) are shown in Table 5.7.

<table>
<thead>
<tr>
<th>Coil</th>
<th>B_{max} (T)</th>
<th>Total Thrust (N)</th>
<th>I_{sp} (s)</th>
<th>η (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>0.125</td>
<td>0.97</td>
<td>990</td>
<td>7.1</td>
</tr>
<tr>
<td></td>
<td>0.19</td>
<td>1.13</td>
<td>1150</td>
<td>7.5</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>1.34</td>
<td>1370</td>
<td>8.9</td>
</tr>
<tr>
<td>C2</td>
<td>0.08</td>
<td>0.85</td>
<td>870</td>
<td>6.9</td>
</tr>
<tr>
<td></td>
<td>0.12</td>
<td>0.98</td>
<td>1000</td>
<td>7.4</td>
</tr>
<tr>
<td></td>
<td>0.16</td>
<td>1.09</td>
<td>1110</td>
<td>7.6</td>
</tr>
<tr>
<td>Self-Field</td>
<td>-</td>
<td>0.61</td>
<td>620</td>
<td>5.0</td>
</tr>
</tbody>
</table>
5.2 Applied-Field MPD Thrusters

Fig. 5.32 Experimental data of AFMPDTs [36]. Data of Myers Ar(100 mg/s) were obtained for similar conditions to the present simulation except for thruster size. Thruster of Myer’s is much larger than simulated thruster. (Simulation Result: $I_{sp}=1,370$ s, $\eta = 8.9\%$).

The total thrust can be divided into each contribution as mentioned above. In Fig. 5.33, respective thrust components for C1 are plotted as a function of the applied magnetic field strength. In all cases, thermal thrust is dominant, and is gradually increased with applied magnetic field. The walls on which thermal thrust acts can be subdivided into four portions; flared nozzle, cathode tip, inlet wall ($z = 0$, $r < 7$, $r > 11$ mm), and wall contacting with the plume region. Each contribution to the thermal thrust $0.65$ N for $B_{max} = 0.25$ T with C1 is shown in Table 5.10. The thrust acting on the inlet wall, and subsequently on the anode, is dominant component in the thermal thrust. The former is enhanced by the intensive Joule heating around the inlet. Intensive heating also causes high pressure at the propellant injection port, hence the $F_{in}$ ultimately also becomes high. The much higher contribution of the anode than that of the cathode is attributed to the characteristic pressure distribution shown in Fig. 5.34, where the pressure on the anode nozzle part is higher than that around the cathode tip due to centrifugal force caused by the swirl motion of the flow. This tendency was experimentally observed by Tahara for 0.4 g/s of Hydrogen [38]. In the experiment, it was shown that the pressure at the cathode tip is decreased as the strength of applied magnetic field is increased under a constant discharge current. However, in the present simulation, the highest pressure at the cathode tip is obtained for $B_{max} = 0.25$ T with C1
probably due to an increase in the radial pinch force.

Among the thruster components in Fig. 5.33, increasing rate of $F_{\text{Hall}}$ is notable, because this component is expected to scale as $B^2$. For $B_{\text{max}} = 0.25$ T with C1, $F_{\text{Hall}}$ amounts to 26% of the total thrust. Figure 5.35-(a) shows the distribution of the axial Lorentz force with regard to $j_\theta$, that is, $-j_\theta B_r$ for $B_{\text{max}} = 0.25$ T with C1. The Lorentz force is highest near the inlet because of the high azimuthal current density of $\mathcal{O}(10^6)$ A/m$^2$ as shown in Fig. 5.36, which is comparable to the discharge current density. There appears a region with negative Lorentz force around the cathode tip at which net radial magnetic field component becomes negative, i.e. $B_r^{\text{ap}} + B_r^{\text{ind}} < 0$ due to diamagnetic effect caused by the induced azimuthal current, although the order of decelerating Lorentz force is much less than the inertial force. The volume integral of $-j_\theta B_r$ inside the thruster ($z < 55$ mm) is 0.2 N and 0.13 N in the plume region ($z > 55$ mm), therefore it can be said that the Hall acceleration mainly occurs in the thruster. On the other hand, in the case of $B_{\text{max}} = 0.16$ T with C2, the $F_{\text{Hall}}$ within the thruster is no more than 0.05 N due to the lower magnetic field strength and growth of region with negative radial magnetic field as shown in Fig. 5.35-(b).

![Fig. 5.33 Thrust components for C1, Ar, 0.1 g/s, $J = 1kA$. Parameters of the coil C1 are shown in Table 5.7.](image_url)
Table 5.10 Components of thermal thrust for $B_{\text{max}} = 0.25 \text{ T}$ with C1, Ar, 0.1 g/s, $J = 1 \text{ kA}$.

<table>
<thead>
<tr>
<th>Component</th>
<th>Thrust (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flared nozzle part</td>
<td>0.22</td>
</tr>
<tr>
<td>Cathode tip</td>
<td>$5 \times 10^{-2}$</td>
</tr>
<tr>
<td>Inlet wall</td>
<td>0.37</td>
</tr>
<tr>
<td>Wall contacting with plume region</td>
<td>$4.3 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Fig. 5.34 Pressure distribution for $B_{\text{max}} = 0.25 \text{ T}$ with C1 (Pa), Ar, 0.1 g/s, $J = 1 \text{ kA}$.
Parameters of the coil C1 are shown in Table 5.7.
Chapter 5 Attempt toward Low-Power Operation Modes

(a) $B_{\text{max}} = 0.25 \text{ T (C1)}$

(b) $B_{\text{max}} = 0.16 \text{ T (C2)}$

Fig. 5.35 Lorenz force $-j\alpha B_z$ distribution ($\text{N/m}^3$), $\text{Ar, 0.1 g/s, } J = 1 \text{ kA}$; (a) $B_{\text{max}} = 0.25 \text{ T with C1}$, (b) $B_{\text{max}} = 0.16 \text{ T with C2}$. Parameters of the coils (C1,C2) are shown in Table 5.7.

Fig. 5.36 Azimuthal current density ($-j\alpha$ is plotted) for $B_{\text{max}} = 0.25 \text{ T with C1}, \text{Ar, 0.1 g/s, } J = 1 \text{ kA}$. Parameters of the coil C1 are shown in Table 5.7.
5.2.6 Energy Conversion Processes

Generally it is believed that the swirl momentum driven by azimuthal Lorenz force can be converted to axial momentum through a solid or a magnetic nozzle. However, detailed energy conversion processes within the thruster have not been clarified. In order to examine the energy conversion process, the variation of the axial energy fluxes of the radial, azimuthal, axial kinetic energy, and the specific enthalpy are shown in Fig. 5.37:

\[
\begin{align*}
\text{Radial} &= \int \frac{1}{2} \rho u_r^2 u_r dS_z, \\
\text{Azimuthal} &= \int \frac{1}{2} \rho u_\theta^2 u_\theta dS_z,
\end{align*}
\]

\[
\begin{align*}
\text{Axial} &= \int \frac{1}{2} \rho u_z^2 dS_z, \\
\text{Specific Enthalpy} &= \int \frac{5}{2} \rho u_z dS_z.
\end{align*}
\]

(5.36)

Inside the thruster, the azimuthal kinetic energy and the specific enthalpy are rapidly increased due to the azimuthal Lorentz force and the Joule heating. It can be seen that the azimuthal kinetic energy begins to rapidly decrease around \( z = 6 \) mm, from which the anode diverges. On the other hand, the specific enthalpy gradually diminishes within the thruster. Comparing the increase in the axial kinetic energy with the decrease in the specific enthalpy and the azimuthal kinetic energy, the increase in the axial kinetic energy between \( z = 18 \) mm and \( z = 55 \) mm is 1.05 kW which is much less than the sum of the decrease in the azimuthal kinetic energy -2.86 kW and specific enthalpy -0.95 kW. Therefore most of the azimuthal kinetic energy and the specific enthalpy are inferred to be lost by friction loss or thermal conduction to the electrodes.

By integrating the heat flux of heavy particles on the electrodes, the thermal losses of the internal energy of heavy particles to the anode and the cathode are estimated to be 4.2 kW and 2.9 kW respectively. Since these are much less than input power (1 kA \( \times \) 81 V = 81 kW), main cause of the energy loss is supposed to be thermal loss with regard to electrons to the anode. Many experiments also indicated that energy loss with regard to electrons is a main energy loss mechanism [16,41,83].

On the other hand, in the plume region from \( z = 55 \) mm to \( z = 100 \) mm, the increase in the axial kinetic energy is 1.20 kW which is very close to the sum of the decrease in the azimuthal kinetic energy -0.39 kW and specific enthalpy -0.95 kW. Thus effective energy conversion from the azimuthal kinetic energy and the specific enthalpy to the axial kinetic energy seems to occur in this region. The apparent saturation of the axial
Chapter 5 Attempt toward Low-Power Operation Modes

kinetic energy flux seen in \( z > 100 \text{ mm} \) is probably due to the radial plasma leakage from the computational domain.

Fig. 5.37 Axial energy fluxes of axial, radial, azimuthal kinetic energy and specific enthalpy for \( B_{\text{max}} = 0.25 \text{ T} \) with C1, Ar, 0.1 g/s, \( J = 1 \text{ kA} \). Parameters of the coil C1 are shown in Table 5.7.

Table 5.11 Comparison of axial energy fluxes at \( z = 18, 55, \) and 100 mm (kW), \( B_{\text{max}} = 0.25 \text{ T} \) with C1, Ar, 0.1 g/s, \( J = 1 \text{ kA} \).

<table>
<thead>
<tr>
<th></th>
<th>( z = 18 \text{ mm} )</th>
<th>( z = 55 \text{ mm} )</th>
<th>( z = 100 \text{ mm} )</th>
<th>Difference between ( z = 55, 18 \text{ mm} )</th>
<th>Difference between ( z = 100, 55 \text{ mm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td>1.16</td>
<td>1.45</td>
<td>3.41</td>
<td>1.05</td>
<td>1.20</td>
</tr>
<tr>
<td>Azimuthal</td>
<td>3.45</td>
<td>0.77</td>
<td>0.23</td>
<td>-2.86</td>
<td>-0.39</td>
</tr>
<tr>
<td>Specific enthalpy</td>
<td>4.15</td>
<td>2.89</td>
<td>2.15</td>
<td>-0.95</td>
<td>-0.95</td>
</tr>
</tbody>
</table>

5.2.7 Remarks for Performance Improvement

As mentioned in subsection 5.2.5, the present thrust efficiency is insufficient for a practical application. Although the working conditions or the thruster configuration are not optimized at this stage, it will be possible to obtain some insights into performance improvement from the simulation results.

Great issue to be resolved is reduction of the thermal loss on the anode. In the present condition, the discharge current concentrates around the inlet part, thus most of the input energy is deposited in the narrow upstream region. Thus the rapidly increased
specific enthalpy of electrons is considerably lost within the thruster. Therefore, the thermal energy should be converted not through a solid nozzle but through a magnetic nozzle. For this purpose, operation with a high magnetic field and a low mass flow rate will be effective to enhance the Hall parameter within the thruster. In addition, swirl motion of the plasma within the thruster seems harmful, because it induces compression of the plasma toward the anode on which significant thermal loss can be inferred. Therefore, an operation with a low swirl motion but a high azimuthal current is thought to be preferable. To reduce the swirl motion, the current concentration around the inlet should be avoided, thus discharge current should be expanded toward downstream. This may be attained by changing electrode configuration, magnetic field configuration, or propellant species.

5.2.8 Summary

A two-dimensional numerical code for the applied-field MPD thruster is developed with the vector potential formulation. This study aims to simulate flowfields of several hundreds of kW level AFMPDT under the condition of a constant mass flow rate of 0.1 g/s with argon propellant, and a constant discharge current of 1 kA. The influences of the applied magnetic field strength and its divergence angle on the flowfields and performances are examined for a thruster with a flared anode and a short cathode. The maximum magnetic field inside thruster is varied from 0.08 to 0.25 T.

With raising the strength of the applied magnetic field, the discharge current path expands toward the downstream region in particular in the plume region, whereas the current on the cathode tends to shift upstream, which implies that the current concentration on the cathode tip can be reduced by magnetic field application. The velocity distribution indicates that the plasma interacts with the magnetic field very well in the plume region.

The main component of the thrust is thermal thrust, and it is found that thermal thrust obtained on the inlet wall and on the anode is dominant. The Hall acceleration amounts to 26% for $B_{\text{max}} = 0.25$ T with rapidly diverging magnetic field, and mainly occurs within the thruster.

From the analysis of the axial energy flux, it is suggested that, in the thruster, most of
the azimuthal kinetic energy and the specific enthalpy is lost as the thermal conduction to the electrodes. On the other hand, in the plume region, effective energy conversion from the azimuthal kinetic energy and the specific enthalpy to the axial kinetic energy occurs.

According to the obtained results, an operation with a low swirl motion and with high azimuthal current is thought to be preferable.
Chapter 6. Conclusions

The plasma flowfields and thrust performances of Magnetoplasmadynamic thrusters are numerically investigated. Two-dimensional numerical codes are developed to simulate the acceleration processes in thermally and chemically nonequilibrium magnetohydrodynamic flowfields. To explore various operation conditions of MPDTs, the detailed physical modeling is conducted to take into account substantial phenomena such as the Hall effect. The present study focuses on three important problems in the research of MPDTs.

A) Detailed understanding of plasma flowfields within a thruster
B) Plasma behavior in a coaxial SFMPDT operated around critical current, and suppression of anode starvation
C) Flowfields and performances with low-power operation modes of a pulsed SFMPDT and an AFMPDT

The first is conducted through the discussion on a two-dimensional SFMPDT under the condition of $m = 1.25 \text{ g/s (Ar)}, J = 8 - 12 \text{ kA}$. From the results, following remarks are obtained.

1. The current path is obliquely skewed in the thruster due to the Hall effect. The distortion of the calculated current path in the vicinity of the flared anode surface excessively appears in the calculation compared to the experimental data. The fan-shaped distribution of the electron number density around a cathode is well captured by the calculation, although quantitative argument remains.
2. The calculated electron temperature is overestimated around the anode surface
compared to experimental data measured by the relative intensity method. To examine this discrepancy, collisional-radiative model of Ar-II is incorporated in the model, and the excitation temperature is numerically estimated from the populations of excited ions. The calculated excitation temperature becomes close to the measured data. This suggests that the plasma deviates from local thermodynamic equilibrium (LTE) near the flared anode surface, while LTE assumption is still valid around the cathode.

3. Calculated thrust and thrust efficiency are overestimated due to uncertainty of a sheath voltage and deviation from the complete two-dimensional uniformity in the actual equipment.

Second topic is examined for a coaxial Self-Field MPDT. By changing the discharge current around the critical current ($\approx 6$ kA) under a constant mass flow rate of 0.8 g/s (Ar), the flowfields operated below and above the critical current are compared to clarify substantial differences. The results demonstrate followings.

1. The Hall parameter on anode surface increases together with the current, leading to an obliquely skewed current profile at discharge currents higher than the critical current.

2. An increase in the radial component of the Lorentz force induces the depletion of the gas density around the anode surface, which corresponds to the shortage of current carriers.

3. To suppress anode starvation due to carrier shortage, segmented anodes are introduced. The result shows that the current-carrier shortage at anode edge is suppressed due to enhancement of the plasma density on the anode and reduction of the current concentration, which implies stabilization of the discharge.

The final argument on low-power operation is divided into two parts; pulsed MPDT and applied-field MPDT.

**Pulsed MPDT**

With current waveforms assumed by a simple function of $J(t)=J_{max}(1-\exp(-t/\tau))$ ($\tau$: discharge time constant), the influences of discharge current waveform and propellant species (Ar and He respectively) on temporal plasma behavior and performances are
Chapter 6 Conclusions

1. Under a constant mass flow rate, specific impulse and thrust efficiency can be improved by shortening the current rising time.

2. The electromagnetic acceleration is dominant for He propellant, which enhances specific impulse for a fixed discharge duration time.

3. Performances comparable to that of steady operation can be achieved with a discharge duration time ($T_{on}$) of 600 µs. Specific impulse and thrust efficiency of He approaches those of steady state faster than Ar.

Applied-Field MPDT

A two-dimensional numerical code to analyze several hundreds of kW level AFMPDT is developed with the vector potential formulation. The influences of the strength of the applied magnetic field, and its divergence angle on the flowfields and performance are examined for a thruster with a flared anode and a short cathode under the condition of $\dot{m} = 0.1 \text{ g/s (Ar)}$, $J = 1 \text{kA}$.

1. With raising the strength of the applied magnetic field, discharge current path expands toward a downstream region in particular in the plume region, whereas the current on the cathode tends to shift upstream. Rapidly diverging magnetic field is effective for smooth acceleration.

2. Total thrust and discharge voltage are linearly increases with applied magnetic field. The main component of the thrust is thermal thrust, in which contributions of inlet wall and flared anode are significant. Hall thrust amounts to 26% of total thrust.

3. The azimuthal kinetic energy and the specific enthalpy are rapidly increased around the inlet, and they are lost to the electrodes via thermal conduction. On the other hand, in the plume region, effective energy conversion from the azimuthal kinetic energy and the specific enthalpy to the axial kinetic energy occurs.

4. An operation with a low swirl motion and with high azimuthal current is thought to be preferable.
Issues in future

Three problems in the research of MPDTs have been discussed in this thesis. A common problem for each subject is how to predict thrust efficiency correctly. For this purpose, discharge voltage including sheath effect has to be estimated with satisfactory accuracy. The difficulty to evaluate discharge voltage is a common problem for any MPDT configurations. Also, energy loss with regard to electrons is thought to play a dominant role on energy loss mechanism. These effects will be vital for correct prediction and optimization of thrust performance. These will be useful to predict time-varying sheath voltage in the calculation of a pulsed MPDT flowfield.

It has been shown that anode starvation (current-carrier shortage) associated with current concentration and plasma density depletion appears at the anode edge above the critical current. However, the current-carrier shortage is supposed to be just a trigger of voltage oscillation, namely onset. Since it is inferred that plasma and electrode interact to each other, sheath effect and anode erosion processes will have to be incorporated in the model to understand the essence of onset phenomenon.

From the result of AFMPDT, it is suggested that an operation with a low swirl motion and with high azimuthal current is preferable. For such a purpose, mass flow rate should be reduced much more, and magnetic field strength must be increased further. However, it is hard to simulate such a condition with the fluid approximation, therefore development of a particle code for AFMPDT flowfield analyses will be desired.
Appendix A

Governing Equations for Coaxial Coordinate System

When the system equations for coaxial coordinate system are multiplied by the volume of the cell $rdrd\theta dz$, Eq. (2.45) is reduced to the following form which is actually used for the analyses of the coaxial MPDTs [20].

$$\frac{\partial rQ}{\partial t} + \frac{\partial rF}{\partial z} + \frac{\partial rG}{\partial r} = G + rS. \quad \text{(A.1)}$$

With this form, the governing equations of (2.7)-(2.12) are reduced to the following forms.

Total mass density

$$\frac{\partial r\rho}{\partial t} + \frac{\partial r\rho u_z}{\partial z} + \frac{\partial r\rho u_r}{\partial r} = 0. \quad \text{(A.2)}$$

Mass density of $i$-fold ion species

$$\frac{\partial r\rho_i}{\partial t} + \frac{\partial r\rho_i u_z}{\partial z} + \frac{\partial r\rho_i u_r}{\partial r} = r\dot{\rho}_i. \quad \text{(A.3)}$$

Momentum

$$\frac{\partial r\rho u_z}{\partial t} + \frac{\partial}{\partial z}\left[r\left(\rho u_z u_r - \frac{B_z B_r}{\mu_0}\right)\right] + \frac{\partial}{\partial r}\left[r\left(\rho u_z^2 + p + \frac{B_z^2}{2\mu_0} - \frac{B^2}{\mu_0}\right)\right]$$

$$= \rho u_z^2 + p + \frac{B^2}{2\mu_0} - \frac{B^2}{\mu_0} + r\left(\frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{z\theta}}{\partial r} + \frac{\tau_{z\theta}}{r} - \frac{\tau_{\theta\theta}}{r}\right). \quad \text{(A.4)}$$

$$\frac{\partial r\rho u_\theta}{\partial t} + \frac{\partial}{\partial z}\left[r\left(\rho u_\theta u_z - \frac{B_\theta B_z}{\mu_0}\right)\right] + \frac{\partial}{\partial r}\left[r\left(\rho u_\theta u_r - \frac{B_\theta B_r}{\mu_0}\right)\right]$$

$$= -\rho u_\theta u_z + \frac{B_\theta B_z}{\mu_0} + r\left(\frac{\partial \tau_{z\theta}}{\partial z} + \frac{\partial \tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r}\right). \quad \text{(A.5)}$$

$$\frac{\partial r\rho u_z}{\partial t} + \frac{\partial}{\partial z}\left[r\left(\rho u_z^2 + p + \frac{B_z^2}{2\mu_0} - \frac{B^2}{\mu_0}\right)\right] + \frac{\partial}{\partial r}\left[r\left(\rho u_z u_r - \frac{B_z B_r}{\mu_0}\right)\right]$$

$$= r\left(\frac{\partial \tau_{z\theta}}{\partial z} + \frac{\partial \tau_{z\theta}}{\partial r} + \frac{\tau_{z\theta}}{r}\right). \quad \text{(A.6)}$$
Appendix A

Internal energy of heavy particles (neutrals and ions)

\[
\frac{\partial r U_h}{\partial t} + \frac{\partial r U_h u_z}{\partial z} + \frac{\partial r U_h u_r}{\partial r} = r \left[ -p_h \nabla \cdot \mathbf{u} + \Phi + \frac{\partial}{\partial z} \left( \lambda_h \frac{\partial T_h}{\partial z} \right) + \frac{\partial}{\partial r} \left( \lambda_z \frac{\partial T_h}{\partial r} \right) + \frac{\lambda_h}{r} \frac{\partial T_h}{\partial r} + \delta E \right].
\] (A.7)

Internal energy of electrons and ionization energy

\[
\frac{\partial r (U_e + U_i)}{\partial t} + \frac{\partial r (U_e + U_i) u_z}{\partial z} + \frac{\partial r (U_e + U_i) u_r}{\partial r} = r \left[ -p_e \nabla \cdot \mathbf{u} + \frac{j^2}{\sigma} + \frac{\partial}{\partial z} \left( \lambda_e \frac{\partial T_e}{\partial z} \right) + \frac{\partial}{\partial r} \left( \lambda_e \frac{\partial T_e}{\partial r} \right) + \frac{\lambda_e}{r} \frac{\partial T_e}{\partial r} + \frac{5k}{2e} j \cdot \nabla T_e \right] - \frac{1}{en_e} j \cdot \nabla p_e - \delta E \right].
\] (A.8)

Induction equation (for azimuthal component of \( B \))

\[
\frac{\partial r B_\theta}{\partial t} + \frac{\partial r (B_\theta u_z - u_\theta B_z)}{\partial z} + \frac{\partial r (B_\theta u_r - u_\theta B_r)}{\partial r} = u_r B_\theta - u_\theta B_r
\]

\[
+ r \left[ \frac{\partial}{\partial z} \left( \frac{1}{\mu_0 \sigma} \frac{\partial B_\theta}{\partial z} \right) + \frac{\partial}{\partial r} \left( \frac{1}{\mu_0 \sigma} \frac{\partial B_\theta}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{B_\theta}{\mu_0 \sigma} \right) - \frac{B_\theta}{\mu_0 \sigma} r^2 \right]
\]

\[
+ \frac{\partial}{\partial z} \left( \frac{\beta' B_\theta}{\mu_0} \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \right) - \frac{\partial}{\partial r} \left( \frac{\beta' B_\theta}{\mu_0} \frac{\partial B_\theta}{\partial z} \right)
\]

\[
- \frac{\partial}{\partial z} \left( \beta' B_\theta \sigma (u_z B_r - B_z u_r) \right)
\]

\[
+ \frac{\partial}{\partial z} \left( \frac{\beta'^2 B_\theta B_z \sigma}{\mu_0} \frac{\partial B_\theta}{\partial r} + \frac{\beta'^2 B_z^2 \sigma}{\mu_0} \frac{\partial B_\theta}{\partial z} \right)
\]

\[
- \frac{\partial}{\partial r} \left( \beta' B_\sigma (u_z B_r - B_z u_r) \right)
\]

\[
+ \frac{\partial}{\partial r} \left( \frac{\beta'^2 B_z^2 \sigma}{\mu_0} \frac{\partial B_\theta}{\partial r} + \frac{\beta'^2 B_z B_\sigma \frac{\partial B_\theta}{\partial z}}{\mu_0} \right)
\]

\[
- \frac{\partial}{\partial z} \left( \beta' \frac{\partial p_e}{\partial r} \right) - \frac{\partial}{\partial r} \left( \beta' \frac{\partial p_e}{\partial z} \right) \right].
\] (A.9)
Here, $\beta^*=1/en_e$. Three components of the magnetic field are taken in the above equations, which can be used for the formulation of AFMPDT.* The effect of electron pressure gradient is included in Eqs. (A.8) and (A.9). In the derivations of the above equations, the mathematical formulas of the divergence for tensor $T_{ij} (=T_{ji})$ are used:

\[
\begin{aligned}
    r \text{ component: } & \quad \frac{\partial T_{rij}}{\partial x_j} = \frac{\partial T_{rz}}{\partial z} + \frac{\partial T_{r}}{\partial r} + \frac{T_{r}}{r} - \frac{T_{\theta\theta}}{r}, \\
    \theta \text{ component: } & \quad \frac{\partial T_{\theta\theta}}{\partial x_j} = \frac{\partial T_{\theta\theta}}{\partial z} + 2 \frac{T_{r\theta}}{r}, \\
    z \text{ component: } & \quad \frac{\partial T_{z}}{\partial x_j} = \frac{\partial T_{z}}{\partial z} + \frac{T_{z}}{r}.
\end{aligned}
\]  

(A.10) (A.11) (A.12)

Since the flow is assumed to be axisymmetric, the derivatives with regard to $\theta$ are to vanish. The viscous tensors for coaxial coordinate system are

\[
\begin{aligned}
    \bar{\tau}_{zz} = \frac{2}{3} \mu \left( \frac{\partial u_z}{\partial z} - \frac{u_r}{r} - \frac{\partial u_r}{\partial r} \right), & \quad \bar{\tau}_{r} = \mu \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right), \\
    \bar{\tau}_{rz} = \frac{2}{3} \mu \left( \frac{\partial u_z}{\partial r} + \frac{u_r}{r} - \frac{\partial u_r}{\partial z} \right), & \quad \bar{\tau}_{r\theta} = \frac{2}{3} \mu \left( \frac{2 u_r}{r} - \frac{\partial u_r}{\partial r} - \frac{\partial u_z}{\partial z} \right), \\
    \bar{\tau}_{r\theta} = \mu \left( \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right), & \quad \bar{\tau}_{\theta\theta} = \mu \frac{\partial u_{\theta}}{\partial z}.
\end{aligned}
\]  

(A.13) (A.14) (A.15)

---

* This is true if $j = \nabla \times (B^{op} + B^{ind})/\mu_0$ which is justified by the equality of $\nabla \times B^{op} = 0$ satisfied in the computational region of an AFMPDT.
Appendix B

Implicit Method for Magnetic Diffusion Term

When a particular source term renders a calculation instable owing to its shorter characteristic time than those for the other physical phenomena, it is preferable to treat the source term implicitly. In the calculation of a pulsed MPDT, only the magnetic diffusion term is semi-implicitly dealt with. In order to maintain the temporal accuracy of second-order, the explicit two-step method for the explicitly treated terms \( R_{\text{exp}} \) is combined with an implicit treatment for \( R_{\text{imp}} \):\[Q_{n}^{a+1} = Q_{n}^{a} + \Delta t R_{\text{exp}} \left( Q_{n}^{a+1/2} \right) + \frac{\Delta t}{2} \left\{ R_{\text{imp}} \left( Q_{n}^{a} \right) + R_{\text{imp}} \left( Q_{n}^{a+1} \right) \right\}. \quad (B.1)\]
The implicitly treated term \( R_{\text{imp}} \) is linearized by using \[R_{\text{imp}} \left( Q_{n}^{a+1} \right) = R_{\text{imp}} \left( Q_{n}^{a+1}, Q_{n}^{a} \right) + \frac{\partial R_{\text{imp}}}{\partial Q_{n}^{a}} \Delta Q_{n}^{a} + O(\Delta t^{2}), \quad (B.2)\]where \( Q_{\text{exp}} \), \( Q_{\text{imp}} \) represent explicitly and implicitly treated variables respectively, and \( \Delta^{n} Q = Q_{n}^{a+1} - Q_{n}^{a} \). For conservative equations including implicitly treated terms, Eq. (B.1) is reduced to \[\left(1 - \frac{\Delta t}{2} \frac{\partial R_{\text{imp}}}{\partial Q_{n}^{a}} \right) \Delta Q_{n}^{a} = \Delta t R_{\text{exp}} \left( Q_{n}^{a+1/2} \right) + \frac{\Delta t}{2} \left\{ R_{\text{imp}} \left( Q_{n}^{a} \right) + R_{\text{imp}} \left( Q_{n}^{a+1}, Q_{n}^{a} \right) \right\}. \quad (B.3)\]Once an equation is written in this delta-form, \( \Delta^{n} Q_{n}^{a} \) can be evaluated with an iterative method for linear algebraic equations.

Generally, a flux for a diffusive term includes spacial derivatives \( Q_{\xi}, Q_{\eta} \), as well as \( Q \):\[\frac{\partial V_{1}(Q, Q_{\xi})}{\partial \xi} + \frac{\partial V_{2}(Q, Q_{\eta})}{\partial \eta} + \frac{\partial V_{3}(Q, Q_{\xi})}{\partial \xi} + \frac{\partial V_{4}(Q, Q_{\eta})}{\partial \eta} \quad (B.4)\]The linearization of the first term can be written as follows:\[\frac{\partial V_{1}^{a+1}}{\partial \xi} = \frac{\partial}{\partial \xi} \left[ V_{1}^{a} + \left( \frac{\partial V_{1}}{\partial Q} \right)^{a} \Delta^{a} Q + \left( \frac{\partial V_{1}}{\partial Q_{\xi}} \right)^{a} \Delta^{a} Q_{\xi} \right] + O(\Delta t^{2}) \quad (B.5)\]
\[= \frac{\partial}{\partial \xi} \left[ V_{1}^{a} + (W_{1} - R_{1,\xi})^{a} \Delta^{a} Q + \frac{\partial}{\partial \xi} (R_{1} \Delta^{a} Q_{\xi} + O(\Delta t^{2}) \right]. \]
where

\[ W_1 = \frac{\partial V_1}{\partial Q}, \quad R_1 = \frac{\partial V_1}{\partial Q_{\xi}}, \quad R_{1,\xi} = \frac{\partial R_1}{\partial \xi}. \]

The forth term in Eq. (B.4) can be written in the same form by replacing \( \xi \) with \( \eta \). As for the cross-derivative terms, that is second and third terms in Eq. (B.4), can be evaluated explicitly without loss of accuracy.

The magnetic diffusion term in the induction equation for \( \psi = rB_\theta \) is given as follows in the dimensionless form:

\[ \frac{\partial}{\partial z} \left( \frac{1}{R_m} \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial r} \left( \frac{1}{R_m} \frac{\partial \psi}{\partial r} \right) - \frac{1}{rR_m} \frac{\partial \psi}{\partial r}. \quad \text{(B.6)} \]

The first and second terms are reduced to the form of Eq. (B.4) via general coordinate transformation. The last first-derivative term is evaluated explicitly in this study. Based on the above discussion, light hand side of Eq. (B.3) for the induction equation with implicitly treated magnetic diffusion term is reduced to the following form:

\[
\left[ 1 + \frac{\Delta t}{2} \left\{ \xi_z \left( \frac{\partial R_{1,\xi}}{\partial \xi} - \frac{\partial^2 R_{1}}{\partial \xi^2} \right) + \xi_r \left( \frac{\partial P_{1,\xi}}{\partial \xi} - \frac{\partial^2 P_{1}}{\partial \xi^2} \right) \right\} + \frac{\Delta t}{2} \left\{ \eta_r \left( \frac{\partial R_{4,\eta}}{\partial \eta} - \frac{\partial^2 R_{4}}{\partial \eta^2} \right) \right. \\
+ \left. \eta_r \left( \frac{\partial P_{4,\eta}}{\partial \eta} - \frac{\partial^2 P_{4}}{\partial \eta^2} \right) \right\} \right] \Delta^n \psi = \text{RHS}, \quad \text{(B.7)}
\]

where

\[ R_1 = \frac{\partial}{\partial \psi_{\xi}} \left( \frac{\xi_{z,\psi_{\xi}}}{R_m} \right) = \frac{\xi_{z}}{R_m}, \quad R_4 = \frac{\partial}{\partial \psi_\eta} \left( \frac{\eta_{z,\psi_\eta}}{R_m} \right) = \frac{\eta_{z}}{R_m}, \]

\[ P_1 = \frac{\partial}{\partial \psi_{\xi}} \left( \frac{\xi_{r,\psi_{\xi}}}{R_m} \right) = \frac{\xi_{r}}{R_m}, \quad P_4 = \frac{\partial}{\partial \psi_\eta} \left( \frac{\eta_{r,\psi_\eta}}{R_m} \right) = \frac{\eta_{r}}{R_m}. \]

In this study, the Generalized Minimal Residual (GMRES) method with restarted technique, which is often denoted as GMRES\((m)\) method, is used to solve the simultaneous linear equation for \( \Delta^n \psi \) of Eq. (B.7) [95]. Then a preconditioning technique is used, i.e. the simultaneous linear equation is transformed into one equivalent to the original equation in order to accelerate convergence. (see Appendix C).
Appendix C

Preconditioning for Linear Algebraic Equation

The simultaneous linear equation of Eq. (B.7) can be written as

\[ Ax = b, \quad (C.1) \]

which is solved generally with an iterative method. Then the convergence rate strongly of an iterative method depends on spectral properties of the coefficient matrix \( A \), hence an iterative method usually involves a second matrix \( M^{-1} \) which transforms the coefficient matrix into one with a more preferable spectrum:

\[ M^{-1} Ax = M^{-1} b. \quad (C.2) \]

An inverse matrix of \( M^{-1} \), that is \( M \), should be one that approximates the original coefficient matrix \( A \). Construction of an efficient iterative solver is equivalent to selecting a favorable preconditioner \( M \). In this study, the matrix \( M \) is evaluated with an incomplete factorization preconditioner given in factored form with \( L \) lower and \( U \) upper triangular. The employed incomplete factorization is

\[ M = (D + L)D^{-1}(D + U), \quad (C.3) \]

where \( D, L \), and \( U \) denote the diagonal matrix, lower and upper triangular matrices with regard to the coefficient matrix \( A \) respectively, i.e. \( A = L + D + U \) [95]. When one incorporates the preconditioning into an iterative method, a residual of \( r = b - Ax \), which is necessary for iterative solvers belonging to the Krylov Subspace methods, is to be replaced with \( r \) determined from the solution of \( Mr = b - Ax \). In this procedure, one can use the following equivalent formulation for \( Mr = y \):

\[ (D + L)z = y, \quad (I + D^{-1}U)r = z. \quad (C.4) \]

Algorithm to solve this system is given by,

Let \( M = (D+L)(I+D^{-1}U) \) and \( y \) be given.

for \( i = 1, 2, \ldots \)

\[ z_i = d_i^{-1}(y_i - \sum_{j \neq i} l_{ij} z_j), \]

for \( i = n, n-1, \ldots \)

\[ r_i = z_i - d_i^{-1} \sum_{j \neq i} u_{ij} r_j. \]
References


References


82. Tahara, H., Kagaya, Y., and Yoshikawa, T., “Exhaust Plume Characteristics of Quasi-Steady MPD Thrusters.” *Proc. of the 27th International Electric Propulsion Con*
References


